Random experiments, sample spaces, and events

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Lecture 1

Learning objectives

After this lecture, we will be able to:

- Give examples of experiments, sample spaces, and events.
- Explain sets and why they are used to describe events.
- Use Venn diagrams to represent events.
- Describe events using set operations.
- Give examples and recognize mutually exclusive events.
- Calculate the cardinality of an event.

Motivation: Monopoly

Is Monopoly a game of luck or strategy? It is your turn and your friends have built hotels *everywhere*. You need to roll two dies and get a 6 or a 7 to avoid paying your friends and declaring bankruptcy. *Everyone* expects you to lose: what are the "chances" you will roll a 6 or a 7, after considering all the scenarios?

In this first lecture, we will introduce and discuss all the necessary definitions in order to be able to quantify risks and chances.

Motivation: A card game

You are playing a card game on a deck with 52 cards of 4 different suits: ♡, ♣, ♢, ♠. You also know that there are 13 cards of each suit. The game is simple: *pick a card, any card*. If that card is red, you win; otherwise, you lose. For the intents of this game, we assume that the order of numbers is 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 followed by three face cards noted as J, Q, K.

Should we play? How should we play? Are there ways to mathematically quantify our risks and our gains?

Definitions

Random experiments

A **random experiment**[1] is defined as an activity or situation where the outcome obtained may be different, even when executed the same

[1] Examples include the flip of a coin (could be Heads or Tails) or rolling a six-sided dice (could get any integer number between 1 and 6.

way. This randomness in the outcome could be due to the inherent nature of the experiment, due to different levels of skill required to get a better result, or due to differences in instrumentation.

What are some other random experiments you can think of? Is measuring the width of a coffee table using a ruler a random experiment? How about cooking? Is a football game an experiment?

Back to the card game

Is our card game a random experiment? In essence, if you always follow the same strategy (e.g., pick the card at the top, or pick the card at the bottom), are you guaranteed the same result?

Sample spaces

With the term **sample space**[2], we refer to the set of all possible outcomes that can be obtained for a random experiment.

A sample space can have a finite or countably infinite number of possible outcomes (e.g., "1, 2, 3, 4, 5, or 6" or any integer number) or it can be an interval of real numbers (e.g., any number between 0 and 1, $[0, 1]$). We call the first type of sample space **discrete**. We will focus on that type of sample spaces in the beginning of the semester. The second type of sample space (where the outcome is a real number belonging to some interval) is called **continuous**. The rest of this lecture is devoted to discrete sample spaces.

[2] In a game of tic-tac-toe, the possible outcomes are win, lose, and tie, whereas in a graded course, the possible outcomes are $A, A-, B+, B, B-, \ldots, F$.

Is food poisoning a possible outcome of cooking? Is snow a possible outcome for tomorrow's weather?

Define the sample space for rolling a die and for rolling two dies. Define the sample space for the distance of any person at any point from the closest McDonald's.

Give an example of a sample space with a finite number of possible outcomes, and an example of a sample space defined over an interval of real numbers.

Back to the card game

Let us think about our card game. The number of outcomes is finite, that is for sure, so our sample space is discrete. *But what is our sample space?* There are multiple ways to describe the sample space here: $S = \{1\heartsuit, 2\heartsuit, \ldots, K\heartsuit, 1\clubsuit, 2\clubsuit, \ldots, K\spadesuit\}$ or $S = \{red, black\}$ or even $S = \{\heartsuit, \clubsuit, \diamondsuit, \spadesuit\}$.

The selection of the proper sample space is guided by what we are trying to achieve. In our motivation, we spoke about the color of the suit, so a sample space of $S = \{red, black\}$ seems the better choice.

Events

The term **event**[3] is used to define a subset of outcomes from the sample space. It can be just one or it can include many of the outcomes. An event can be a combination of outcomes ("get a 4 or more in a six-sided die") or the negation of an outcome ("no rain"). An event is *simple* if it has one outcome ("get dealt a Queen of $\clubsuit$ in a deck of cards") or *compound* if it includes multiple outcomes ("don't lose" implies a win or a tie).

[3] For a student taking a graded class, an event can be to *pass* or to *get a grade higher than or equal to a B.*

Define some events for tomorrow's weather. Is "less than 10 minutes" a possible event for the experiment of counting the time until the next bus arrives? Is "farther than 10 miles" an event for the closest gas station?

In a board game where players roll two six-sided dies, is "getting a 10" a simple or a compound event?

Back to the card game

Let us return to the card game from our motivation. Assume that

$$S = \{1\heartsuit, 2\heartsuit, \dots, K\heartsuit, 1\clubsuit, 2\clubsuit, \dots, K\spadesuit\},$$

then the event E "picking a red card" is a **compound event** as there are 26 outcomes that satisfy it and

$$E = \{1\heartsuit, 2\heartsuit, \dots, K\heartsuit, 1\diamondsuit, \dots, K\diamondsuit\}.$$

Had we picked that $S = \{red, black\}$, then the event E "picking a red card" is a **simple event** as there is only one outcome that satisfies it (note that $E = red$ in this case).

Sets and set operations

Set operations

Set operations are a very useful way to describe events based on several outcomes. The most common set operations (and the ones we will predominantly use in this class) are:

- The union of two events E_1, E_2 as $\boxed{E_1 \cup E_2}$ [4].
- The intersection of two events E_1, E_2 as $\boxed{E_1 \cap E_2}$ [5].
- The complement of an event E as $\boxed{\overline{E}}$ (sometimes is also written as E^c or E') [6].
- The relative complement (sometimes termed as the *difference*) of an event E_2 from event E_1 as $\boxed{E_1 \setminus E_2}$ [7].

[4] *Either* event E_1 *or* E_2 (or *both*!) should happen.

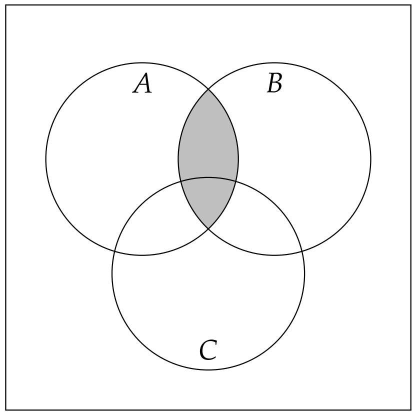
[5] *Both* events E_1 *and* E_2 should happen.

[6] *Any* other event *but* E.

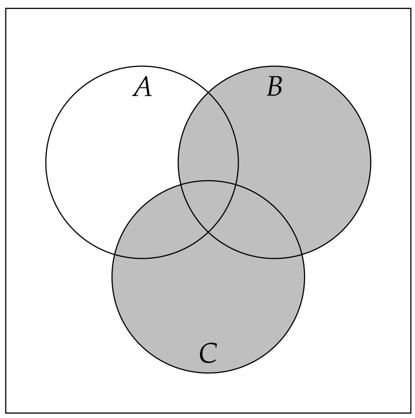
[7] *All outcomes in* E_1 *that are not also in* E_2.

Set operations are nicely described through the use of Venn diagrams. Consider the following examples, where the whole sample space is A, B, and C.

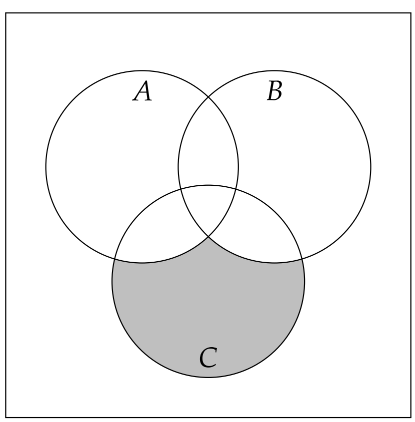
1. A and B should both happen $\to A \cap B$:



2. B or C should happen $\to B \cup C$:



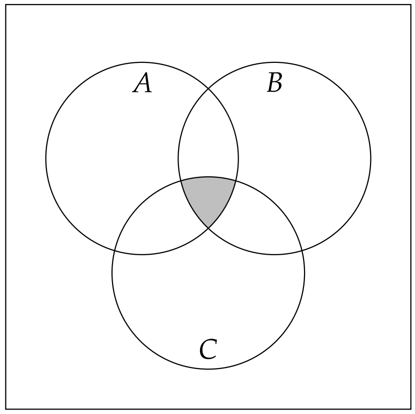
3. Neither A nor B should happen $\to \overline{A} \cap \overline{B}$: [8]



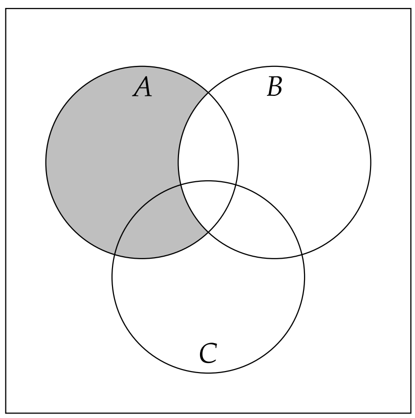
4. A, B, and C should all happen $\to A \cap B \cap C$:

[8] This can also be expressed as:

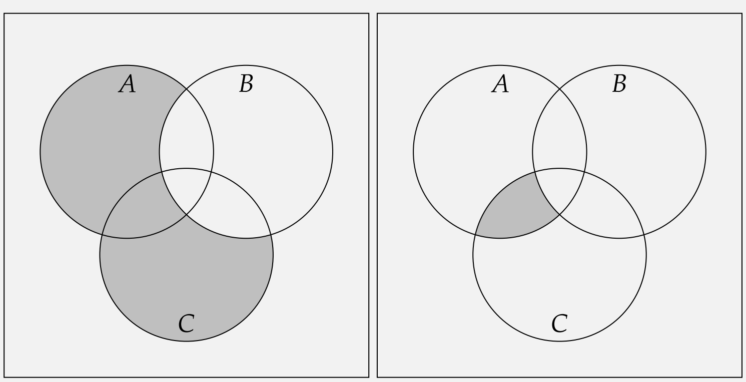
- Only C should happen but not A nor $B \to C \setminus (A \cup B)$.
- A or B should not happen $\to \overline{(A \cup B)}$.



5. A but not B should happen $\to A \setminus B$:



Describe mathematically and in English these two diagrams.



Some more definitions about sets:

- A set that contains no elements (outcomes) is called an empty or a null set, and is denoted by $\emptyset$.

- The sample space is a set containing all outcomes and is typically denoted by S.
- We say that event E_1 is a subset of event E_2 if all outcomes of event E_1 are included in event E_2 [9].
  - We denote this as $E_1 \subseteq E_2$.
  - By definition, $\emptyset \subseteq S$.

[9] In English, this also implies that event E_1 happening immediately signals that event E_2 is happening, too.

In the Venn diagrams earlier, we had $S = A \cup B \cup C$, as these were the only three possible outcomes.

Give an example of a two events where one is a subset of the other.

Finally, we say that two events are **mutually exclusive** [10] if they contain no common outcomes. Mathematically, two events E_1, E_2 are mutually exclusive if

$$E_1 \cap E_2 = \emptyset.$$

[10] You cannot both get a B in a class and $fail$ the class at the same time.

Give an example of a pair of mutually exclusive events.

Back to the card game

Assume that

$$S = \{1\heartsuit, 2\heartsuit, \ldots, K\heartsuit, 1\clubsuit, 2\clubsuit, \ldots, K\spadesuit\},$$

and consider three events:

- $A =$"get a card with the value 3 or less"
- $B =$"get a red card"
- $C =$"get a face card"

What is:

- the union of A and B?

  $A \cup B$: "get a card with the value of 3 or less or a red card."
  - The event happens if we get $7\heartsuit$.
  - The event happens if we get $2\diamondsuit$.
  - The event happens if we get $1\spadesuit$.
  - The event does not happen if we get $10\clubsuit$.

Back to the card game

What is:

- the intersection of B and C?

  $B \cap C$: "get a red and face card."

  - The event happens if we get $Q\heartsuit$.
  - The event happens if we get $J\diamondsuit$.
  - The event does not happen if we get $1\spadesuit$.
  - The event does not happen if we get $K\clubsuit$.

- the intersection of A and C?

  $B \cap C$: "get face card that is less than or equal to 3 in value."

  - The event never happens.
  - In set notation, we have $B \cap C = \emptyset$.
  - B and C are mutually exclusive events.

- the complement of C?

  $\overline{C}$: "not get a face card."

  - The event does not happen if we get $Q\heartsuit$.
  - The event does not happen if we get $J\diamondsuit$.
  - The event happens if we get $1\spadesuit$.
  - The event happens if we get $6\clubsuit$.

Set operation laws

Assume S is the sample space, and A, B, C are some events. Then:

1. $A \cup \overline{A} = S, \quad A \cap \overline{A} = \emptyset, \quad \overline{\overline{A}} = A.$
2. $A \cup B = B \cup A$ and $A \cap B = B \cap A$.
3. De Morgan's laws:
   - $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$.
   - $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$.
4. $A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C.$
5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$

Cardinality

The **cardinality** of an event E [11] is the number of outcomes that it contains and it is denoted by $|E|$. Some important cardinality properties are:

[11] In a graded course (where $S = \{A, A-, B+, B, B-, \ldots F\}$, the cardinality of $E = grade \geq B$ is 4 ($B, B+, A-$, and A), that is $|E| = 4$.

- $E = \emptyset \implies |E| = 0$.
- If E_1 is a subset of E_2, then $|E_1| \leq |E_2|$.
- If events E_1, E_2 are mutually exclusive, then
  - $|E_1 \cap E_2| = 0$.
  - $|E_1 \cup E_2| = |E_1| + |E_2|$.
- For *any* two events E_1, E_2, then
  - $|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$ [12].

[12] Proving this is part of your worksheet for the day.

Back to the card game

Let us finally discuss the actual problem of our motivation. Assume, once again, that our sample space is defined as:

$$S = \{1\heartsuit, 2\heartsuit, \ldots, K\heartsuit, 1\clubsuit, 2\clubsuit, \ldots, K\spadesuit\}.$$

Our game states that we win if we pick a red card. There are 13 $\heartsuit$ and 13 $\diamondsuit$ cards in the game. This gives us a cardinality of 26 outcomes. Recall that in total, our sample space consists of 52 outcomes, that is $|S| = 52$. One might want to reason then that we have 26 favorable outcomes in a total of 52 outcomes...

A class at the University of Illinois at Urbana-Champaign is taught by three different professors. The number of students that took the class and the grades they received are shown in the following table.

Letter Grade	Professor 1	Professor 2	Professor 3	Total
A	108	20	30	158
B	44	49	46	139
C	11	15	15	41
D	0	1	8	9
Total	163	85	99	347

- How many students received an A in Professor 1's class?
- How many students were in Professor 1's class or got an A?
- Are the students who got an A and the students who got a B in Professor 1's class mutually exclusive events?
- How many students got an A but were not in Professor 1's class?

Back to Monopoly

In the beginning of this lecture, we only needed to roll a 6 or a 7 to "survive" another round (so to speak). Let us finish this lecture with the following thought process:

1. The sample space of rolling two dies is:

$$S = \{(1,1), (1,2), (1,3), \ldots, (1,6), (2,1), \ldots, (6,6)\}.$$

2. Counting all possible outcomes, we have that $S = |36|$.
3. The event "roll a 6" contains 5 outcomes:

$$\{(1,5), (2,4), (3,3), (4,2), (5,1)\}.$$

4. The event "roll a 7" contains 6 outcomes:

$$\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}.$$

5. "Roll a 6" and "Roll a 7" are mutually exclusive, hence we have a total of

$$5 + 6 = 11 \text{ favorable outcomes.}$$

6. One could again argue that our "chances" albeit small are not *that small* after all..