Joint distributions: extensions

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Lecture 12

Learning objectives

After these lectures, we will be able to:

- Calculate and use the expectation and variance of jointly distributed random variables.
- Define, calculate, and use the conditional expectation.
- Recognize independence in joint distributions.
- Use independence to calculate probabilities.
- Quantify the level of dependence using covariance and correlation.
- Calculate the expectation and variance of multiple random variables, independent or not.

Motivation: What should I expect?

Consider two jointly distributed random variables (X, Y). What should I expect X to be? What should I expect Y to be? What should I expect X to be if I already know what Y is? Does that change or does it stay the same?

Motivation: Dependence

Knowing whether the value of X affects Y or not is important. Consider, for example, a movie studio planning a series of superhero movies. If the first movie is unsuccessful, and is panned by critics and the audience, then the studio may want to rethink the sequel and subsequent movies. We would want to know the level of dependence between two random variables to help us make decisions better.

Expectations and variances

Once again, we have discussed expectations and variances before; however, those were in the setting of single random variables. Here, we generalize in two or more random variables. As a motivating example, consider a student taking two classes: the student may be interested in the expected grade in one of the two classes alone.

This may ring a bell. Recall that during our last lecture, we discussed the **marginal pmf/pdf** of jointly distributed random variables (X, Y). We specifically said that they come in play when we want to answer the question "what is the probability that X takes a certain value, regardless of Y?" Well, we will use this to calculate expectations and variances. Specifically, we want to answer the questions:

1. "what is the expected value that X takes, regardless of Y?"
2. "what is the variance of X, regardless of Y?"

Of course, both are easily adapted for Y (regardless of X).

Let (X, Y) be two jointly distributed random variables with marginal pmf/pdf $f_X(x)$ and $f_Y(y)$. Then:

$$\begin{array}{llll} & \text{Discrete} & \text{Continuous} & \\ E[X] & = \sum_x x f_X(x) & = \int_{-\infty}^{+\infty} x f_X(x) dx & = \mu_X \\ Var[X] & = \sum_x x^2 f_X(x) - \mu_X^2 & = \int_{-\infty}^{+\infty} x^2 f_X(x) dx - \mu_X^2 & = \sigma_X^2 \\ E[Y] & = \sum_y y f_Y(y) & = \int_{-\infty}^{+\infty} y f_Y(y) dy & = \mu_Y \\ Var[Y] & = \sum_y y^2 f_Y(y) - \mu_Y^2 & = \int_{-\infty}^{+\infty} y^2 f_Y(y) dy - \mu_Y^2 & = \sigma_Y^2 \end{array}$$

Applying the formulae: a chemical mixture

Last time we saw a chemical mixture problem with the volumes of two materials. Here, X and Y are continuous random variables between 0 and 1 representing the material volumes with joint pdf $f_{XY}(x, y) = \frac{2}{5}(2x + 3y)$. Recall that last time we calculated the marginal pdf for X and Y as $f_X(x) = \frac{1}{5}(4x + 3)$ and $f_Y(y) = \frac{6y+2}{5}$.

1. What is the expectation and the variance of the volume of the first material?
2. What is the expectation and the variance of the volume of the second material?

Applying the formulae: a chemical mixture

1. What is the expectation and the variance of the volume of the first material?

   Recall that $f_X(x) = \frac{1}{5}(4x+3)$:

$$E[X] = \int_0^1 x \cdot \frac{1}{5}(4x+3)\,dx = \frac{17}{30} = 0.566\ldots$$

$$Var[X] = \int_0^1 x^2 \cdot \frac{1}{5}(4x+3)\,dx - (E[X])^2 = \frac{71}{900} = 0.0788\ldots$$

2. What is the expectation and the variance of the volume of the second material?

   Recall that $f_Y(y) = \frac{6y+2}{5}$:

$$E[Y] = \int_0^1 y \cdot \frac{6y+2}{5} dy = 0.6$$

$$Var[Y] = \int_0^1 y^2 \cdot \frac{6y+2}{5} dy - (E[Y])^2 = \frac{11}{150} = 0.073\ldots$$

Practice with the following joint distributions:

- Discrete: $f_{XY}(x,y) = \frac{x \cdot y}{18}$, $x = 1,2$, $y = 1,2,3$.
- Continuous: $f_{XY}(x,y) = x + y$, $0 \leq x \leq 1, 0 \leq y \leq 1$.

Conditional expectations and variances

Conditional means and variances can also be defined in the case of two jointly distributed random variables X, Y. They would answer the question: "what should I expect X to be if I know that Y is equal to y?" Clearly, the same question can be asked for Y.

The conditional mean and variance of random variable X given a value for random variable $Y = y$ and of random variable Y given $X = x$ are:

	Discrete	Continuous	
$E[X\|y]$	$=\sum_x x f_{X\|y}(x)$	$=\int_{-\infty}^{+\infty} x f_{X\|y}(x)dx$	$=\mu_{X\|y}$
$Var[X\|y]$	$=\sum_x x^2 f_{X\|y}(x)-\mu^2_{X\|y}$	$=\int_{-\infty}^{+\infty} x^2 f_{X\|y}(x)dx-\mu^2_{X\|y}$	$=\sigma^2_{X\|y}$
$E[Y\|x]$	$=\sum_y y f_{Y\|x}(y)$	$=\int_{-\infty}^{+\infty} y f_{Y\|x}(y)dy$	$=\mu_{Y\|x}$
$Var[Y\|x]$	$=\sum_y y^2 f_{Y\|x}(y)-\mu^2_{Y\|x}$	$=\int_{-\infty}^{+\infty} y^2 f_{Y\|x}(y)dy-\mu^2_{Y\|x}$	$=\sigma^2_{Y\|x}$

Back to the chemical mixture

Recall that we had calculated during the previous lecture that $f_{X|y}(x) = \frac{4x+6y}{6y+2}$.

What is the expectation and the variance of the volume of the first material, given that the second material's volume is equal to 0.6?

We have:

$$E[X|y] = \int_0^1 x\frac{4x+6\cdot 0.6}{6\cdot 0.6+2}dx = 0.5595$$

$$Var[X|y] = \int_0^1 x^2\frac{4x+6\cdot 0.6}{6\cdot 0.6+2}dx - 0.5595^2 = 0.0799.$$

Find the conditional distributions of the two joint distributions below.

- Discrete: $f_{XY}(x,y) = \frac{x\cdot y}{18}$, $x = 1,2$, $y = 1,2,3$.
- Continuous: $f_{XY}(x,y) = x+y$, $0 \leq x \leq 1, 0 \leq y \leq 1$.

Then, find:

- For the first one (the discrete distribution): $E[X|Y=2], Var[X|Y=2]$.
- For the second one (the continuous distribution): $E[Y|X=0.5], Var[Y|X=0.5]$.

Expectations of functions

Finally, as far as expectations are concerned, we take a look at the expectation of a function. Let $h(X, Y)$ be a function of two jointly distributed random variables X, Y. Very similarly to what we did for expectations of functions of single random variables, we get: [1]

$$\text{discrete :} \qquad E\left[h(X,Y)\right] = \sum_x \sum_y h(x,y) f_{XY}(x,y)$$

$$\text{continuous :} \qquad E\left[h(X,Y)\right] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(x,y) f_{XY}(x,y) dxdy$$

[1] Recall that for random variable X and function $g(X)$, we have:

$$\text{discrete :} \qquad E\left[g(X)\right] = \sum_x g(x)p(x)$$

$$\text{continuous :} \qquad E\left[g(X)\right] = \int_{-\infty}^{+\infty} g(x)f(x)dx$$

Again with the chemical mixture

The chemical mixture's quality is evaluated by function $h(X, Y) = 3X + 7Y$. We note here that material 2 is preferable to material 1, and hence the quality is severely favored when material 2 has higher volume. The maximum possible quality is equal to 10 (when both material volumes are equal to 1): in general, the quality ranges from 0 to 10.

What is the expected mixture quality?

We can calculate this as:

$$\begin{aligned} E\left[h(X,Y)\right] &= \int_0^1 \int_0^1 (3x+7y)\,\frac{2}{5}\,(2x+3y)\,dxdy = \\ &= \int_0^1 \frac{1}{5}\left(42y^2 + 23y + 4\right) dy = \\ &= 5.9. \end{aligned}$$

Independence

Independence is a fundamental property of events and random variables. In the past [2] we have discussed independence for events: events A and B are independent if

$$P(A|B) = P(A) \;\text{ or }\; P(A \cap B) = P(A) \cdot P(B).$$

[2] See Lecture 3.

This property proved very useful for calculating basic probabilities. Now, we extend its definition to jointly distributed random variables.

Definition 1 (Independence for random variables) *Two random variables X, Y are independent if any of the following statements hold:*

1. $f_{XY}(x, y) = f_X(x) f_Y(y), \forall x, y$
2. $f_{X|y}(x) = f_X(x), \forall x, y$
3. $f_{Y|x}(y) = f_Y(y), \forall x, y$
4. $P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B), \forall A, B$

In English, the four statements say the same thing; but they do it in different ways. The first one claims that two independent random variables will see their joint pmf/pdf be equal to the product of the individual marginal pmfs/pdfs. **This is typically the easiest way to show (or not) independence of two random variables.** If we find the marginal pmf/pdf of X and X and their product does not always equate to their joint pmf/pdf, then the two variables are not independent.

The second and the third statements are similar to the first definition of independence of events. In essence, they claim that the conditional pmf/pdf of one random variable given the other is equal to the marginal pmf/pdf.

The last statement is interesting, but it needs to be shown for any two sets of values A and B. It states that the probability of both X belonging to set A and Y belonging to set B can be found through the product of the individual probabilities. The last statement is very similar to the first, but instead focuses on sets of values rather than the pmf/pdf.

Discrete random variable independence

Consider two discrete random variables with joint pmf $f_{XY}(x, y) = e^{-3} \frac{2^x}{x! \cdot y!}$ for $x, y \geq 0$. Are random variables X and Y independent?

This looks very intimidating, but we can use some well-known facts from calculus to obtain the answer. Recall that $\sum_{i=0}^{\infty} \frac{\alpha^i}{i!} = e^{\alpha}$. This will be useful.

Now, to find the marginal pmfs:

1. $f_X(x) = \sum_{y=0}^{\infty} e^{-3} \frac{2^x}{x! \cdot y!} = e^{-3} \frac{2^x}{x!} \cdot e = e^{-2} \frac{2^x}{x!}$.
2. $f_Y(y) = \sum_{x=0}^{\infty} e^{-3} \frac{2^x}{x! \cdot y!} = e^{-3} \frac{1}{y!} \cdot \sum_{x=0}^{\infty} \frac{2^x}{x!} = e^{-3} \frac{1}{y!} \cdot e^2 = e^{-1} \frac{1}{y!}$.

We then observe that $f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$, showing independence.

We could have used statements 2 and 3 to show the same thing!

Discrete random variable independence

Consider two discrete random variables with joint pmf $f_{XY}(x, y) = e^{-3}\frac{2^x}{x! \cdot y!}$ for $x, y \geq 0$. Are random variables X and Y independent?

We could first find the conditional pmfs:

1. $f_{X|y}(x) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{e^{-3}\frac{2^x}{x! \cdot y!}}{e^{-1}\frac{1}{y!}} = e^{-2}\frac{2^x}{x!} = f_X(x)$.
2. $f_{Y|x}(y) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{e^{-3}\frac{2^x}{x! \cdot y!}}{e^{-2}\frac{2^x}{x!}} = e^{-1}\frac{1}{y!} = f_Y(y)$.

We then observe that $f_{X|y}(x) = f_X(x)$ and $f_{Y|x} = f_Y(y)$, showing independence again.

The last statement would prove a little tougher, as we would need to show that it is true for any pair of sets of values. Let's see an example where independence does not hold.

Continuous random variable independence

Consider two continuous random variables with joint pdf $f_{XY}(x, y) = x + y, 0 \leq x \leq 1, 0 \leq y \leq 1$. Are the random variables independent?

Similarly to the previous example, let's calculate the marginal pdfs:

1. $f_X(x) = \int\limits_{y=0}^{1} (x + y)\, dy = xy + \frac{y^2}{2}\Big|_0^1 = x + \frac{1}{2}$.
2. Similarly, $f_Y(y) = y + \frac{1}{2}$.

We now note that $f_X(x) \cdot f_Y(y) = x \cdot y + \frac{1}{2}x + \frac{1}{2}y + \frac{1}{4}$, which does not reveal independence, as the product is not always equal to $x + y$. At this point we may claim that the two events are not independent. The same can be said using statements 2 and 3:

1. $f_{X|y}(x) = \frac{x+y}{x+\frac{1}{2}}$.
2. Similarly, $f_{Y|x}(y) = \frac{x+y}{y+\frac{1}{2}}$.

Again, these two are not necessarily equal to $f_X(x)$ or $f_Y(y)$, respectively.

Last, let us consider statement 4: if we are able to find at least one pair of values A, B such that $P(X \in A, Y \in B) \neq P(X \in A) \cdot P(Y \in B)$ should be enough to disprove independence.

Continuous random variable independence

Consider two continuous random variables with joint pdf $f_{XY}(x, y) = x + y, 0 \leq x \leq 1, 0 \leq y \leq 1$. Are the random variables independent?

Consider $A = [0, 0.25]$ and $B = [0.5, 1]$. We have:

$$\begin{aligned} P(X \in A, Y \in B) &= \int_0^{0.25} \int_{0.5}^{1} (x + y)\, dy dx = \\ &= \int_0^{0.25} \left(0.5x + \frac{3}{8}\right) dx = 0.109375. \end{aligned}$$

On the other hand:

1. $P(X \in A) = \int_0^{0.25} f_X(x) dx = \int_0^{0.25} \left(x + \frac{1}{2}\right) dx = 0.15625.$
2. $P(Y \in B) = \int_{0.5}^{1} f_Y(y) dy = \int_{0.5}^{1} \left(y + \frac{1}{2}\right) dy = 0.625.$

We observe that $P(X \in A) \cdot P(Y \in B) = 0.15625 \cdot 0.625 = 0.09765625$ which is not equal to $P(X \in A, Y \in B) = 0.109375$. Hence, the two random variables are not independent.

Alright, so we are able to tell if two random variables are independent or not. Another useful metric though, would be to be able to tell **how dependent** two random variables are.

Covariance

Definition 2 (Covariance) *Covariance is a measure of the* association *between two random variables. For two random variables X and Y, we define* covariance *as:*

$$\sigma_{XY} = Cov\left[X, Y\right] = E\left[(X - E\left[X\right]) \cdot (Y - E\left[Y\right])\right] = E\left[XY\right] - E\left[X\right] \cdot E\left[Y\right].$$

Remember the definition of variance for a single random variable? It was:

$$\sigma_X^2 = Var\left[X\right] = E\left[(X - E\left[X\right])^2\right] = E\left[X^2\right] - (E\left[X\right])^2.$$

This is very similar to the definition of covariance, extended to include jointly distributed random variables.

A few observations that we may make based on the definition follow:

1. If $X \geq E[X]$ whenever $Y \geq E[Y]$ and if $X \leq E[X]$ whenever $Y \leq E[Y]$, then the covariance will be positive.

2. If $X \geq E[X]$ whenever $Y \leq E[Y]$ and if $X \leq E[X]$ whenever $Y \geq E[Y]$, then the covariance will be negative.

3. Finally, and very importantly: **two independent random variables X, Y will have $\sigma_{XY} = Cov[X, Y] = 0$. The inverse is not necessarily true.**

A small Florida example

In Gainesville, FL, summer days are classified as either sunny or rainy. Whenever it is sunny, Floridians go to watch a local baseball team; whenever it is rainy, they tend to forget their umbrellas and they need to buy one. If it is rainy, profits skyrocket for an umbrella selling grocery store and they make $4,500; at the same time, the local team only makes $1,000. If it is sunny, the grocery store only makes $500; the team though makes $2,500 from tickets. Summers are sunny in Florida 65% of the time, and rainy the remaining 35%. What is the covariance of the two company profits?

Let U be the umbrella profits, and T the team profits. To help us collect all data we may construct a small table as follows:

	Sunny	Rainy
Probability	0.65	0.35
Team (T)	$2500	$1000
Umbrellas (U)	$500	$4500

- First, calculate the expected profits:

$$E[U] = 0.65 \cdot 500 + 0.35 \cdot 4500 = \$1900$$
$$E[T] = 0.65 \cdot 2500 + 0.35 \cdot 1000 = \$1975.$$

A small Florida example

- Now, on to calculating $(X - E[X]) \cdot (Y - E[Y])$:
  - when it is sunny:

$$(U - E[U]) \cdot (T - E[T]) =$$
$$= (500 - 1900) \cdot (2500 - 1975) = -805000.$$

  - when it is rainy:

$$(U - E[U]) \cdot (T - E[T]) =$$
$$= (4500 - 1900) \cdot (1000 - 1975) = -2340000.$$

- Last, calculate the covariance:

$$\begin{aligned} Cov(U, T) = E[(U - E[U]) \cdot (T - E[T]) = (4500 - 1900)] &= \\ = 0.65 \cdot (-805000) + 0.35 \cdot (-2340000) &= \\ = -1342250. \end{aligned}$$

Covariance is easier calculate as $E[X \cdot Y] - E[X] \cdot E[Y]$ when the probabilities are given in joint pmf/pdf format.

Covariance for continuous random variables

Earlier, we saw an example of two jointly distributed continuous random variables X and Y with pdf $f_{XY}(x, y) = x + y, 0 \leq x \leq 1, 0 \leq y \leq 1$. We actually calculated their marginal pdfs:

- $f_X(x) = x + \frac{1}{2}$.
- $f_Y(y) = y + \frac{1}{2}$.

We also found that these two are not independent. What is their covariance?

Covariance for continuous random variables

Recall that we can calculate the covariance of X and Y as $E[XY] - E[X] \cdot E[Y]$.

- $E[X] = \int_0^1 x \left(x + \frac{1}{2}\right) dx = \frac{7}{12}$.
- $E[Y] = \int_0^1 y \left(y + \frac{1}{2}\right) dy = \frac{7}{12}$.
- Also: $E[XY] = \int_0^1 \int_0^1 xy(x+y)\, dydx = \frac{1}{3}$.

Finally:

$$\sigma_{XY} = \frac{1}{3} - \frac{7}{12} \cdot \frac{7}{12} = -\frac{1}{144}.$$

Correlation

The problem with covariance is that it is not normalized. A very big covariance or a very small covariance do not necessarily imply the actual level of dependence. This is why we introduce **correlation**, a measure that directly relates its value to the magnitude of dependence.

Definition 3 (Correlation) *Correlation is a measure of the linear relationship between two random variables X and Y. It is calculated as:*

$$\rho_{XY} = \frac{Cov[X,Y]}{\sqrt{Var[X]}\sqrt{Var[Y]}} = \frac{\sigma_{XY}}{\sigma_X \cdot \sigma_Y}.$$

By definition, $-1 \leq \rho_{XY} \leq 1$.

Notice that the numerator of correlation is the covariance itself, normalized by the product of the individual standard deviations (square roots of the variances). A few observations we can make based on this definition.

1. $\rho_{XY} = 0$ implies that X and Y are not correlated.
   - When X and Y are independent then $\sigma_{XY} = \rho_{XY} = 0$.
2. $\rho_{XY} = 1$ implies that X and Y are fully positively correlated.
3. $\rho_{XY} = -1$ implies that X and Y are fully negatively correlated.

Back to Florida

What is the correlation in the Florida example?

First, we calculate individual variances:

- $Var[U] = 0.65 \cdot (500 - 1900)^2 + 0.35 \cdot (4500 - 1900)^2 = 3640000$.
- $Var[T] = 0.65 \cdot (2500 - 1975)^2 + 0.35 \cdot (1000 - 1975)^2 = 508471.25$.

Since we already calculated $Cov(U, T) = -1342250$, we can compute the correlation as:

$$\rho_{UT} = \frac{-1342250}{\sqrt{3640000} \cdot \sqrt{508471.25}} = -0.9866.$$

Hence, as expected, the two profits are almost totally negatively correlated!

Correlation for continuous random variables

Another example we saw earlier: $f_{XY}(x, y) = x + y, 0 \leq x, y \leq 1$ with:

- $f_X(x) = x + \frac{1}{2}$.
- $f_Y(y) = y + \frac{1}{2}$.
- $E[X] = \frac{7}{12}$.
- $E[Y] = \frac{7}{12}$.
- $\sigma_{XY} = -\frac{1}{144}$.

What is the correlation?
First, we find the variances:

$$Var[X] = \int_0^1 x^2 f_X(x)dx - (E[X])^2 = \int_0^1 x^2 \left(x + \frac{1}{2}\right) dx - \left(\frac{7}{12}\right)^2 =$$
$$= \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}.$$

We may similarly calculate $Var[Y] = \frac{11}{144}$, too. Finally:

$$\rho_{XY} = \frac{-\frac{1}{144}}{\sqrt{\frac{11}{144}} \cdot \sqrt{\frac{11}{144}}} = -\frac{1}{11}.$$

When x and y restrict each other

This serves as more of a reminder from calculus. When taking the integral of more than one variable at the same time, we need to be very careful with the bounds we are using.

Let us see this with an example. Assume (X, Y) are two jointly distributed continuous random variables with joint pdf equal to $f_{XY}(x, y) = 3 \cdot (x + y)$. Moreover, assume that $X, Y \geq 0$ and (and this is important!) $X + Y \leq 1$.

Note how the value of random variable X affects the range of values that Y is allowed to take; and vice versa. We need to be very careful with how we proceed in this case. There are three things we need to be able to do.

1. Calculate probability for **both random variables at the same time** and expectations for a function of both random variables.

2. Calculate marginal distributions **for one random variable at a time**.

3. Calculate probabilities, expectations, and variances **for one random variable (forgetting the other exists)** and calculate conditional probabilities, expectations, and variances **for one random variable setting the other equal to a value**.

Let us specifically focus on these three items for the pdf provided: $f_{XY}(x, y) = 3 \cdot (x + y)$ for $x, y \geq 0$, such that $x + y \leq 1$.

Both at the same time

Typical questions:

- Verify that $f_{XY}(x, y)$ is a valid pdf.
- What is the probability that $X \leq 0.3$ and $Y > 0.5$?

To answer these questions, we need to allow x and y to consider all values they can get. For example, if x is allowed to go from 0 to 1, then y is allowed to go from 0 to $1 - x$. On the other hand, if y is allowed to go from 0 to 1, then x is only allowed to go from 0 to $1 - y$.

What we could do wrong: we could possibly allow both x and y to go from 0 to 1, allowing $x + y$ to potentially go higher than 1, breaking the requirement.

Both at the same time

Finally:

- Verify that $f_{XY}(x,y)$ is a valid pdf.

$$\int_0^1 \int_0^{1-x} 3 \cdot (x+y)\, dydx = \int_0^1 3 \cdot \left(xy + \frac{y^2}{2} \right) \Bigg|_0^{1-x} dx =$$
$$= \int_0^1 3 \cdot \left(\frac{1}{2} - \frac{x^2}{2} \right) dx = \left(\frac{3}{2}x - \frac{1}{2}x^3 \right) \Bigg|_0^1 = 1,$$

or

$$\int_0^1 \int_0^{1-y} 3 \cdot (x+y)\, dxdy = \int_0^1 3 \cdot \left(\frac{x^2}{2} + yx \right) \Bigg|_0^{1-y} dy =$$
$$= \int_0^1 3 \cdot \left(-\frac{y^2}{2} + \frac{1}{2} \right) dy = \left(-\frac{1}{2}y^3 + \frac{3}{2}y \right) \Bigg|_0^1 = 1.$$

- What is the probability that $X \leq 0.3$ and $Y > 0.5$?

$$\int_0^{0.3} \int_{0.5}^{1-x} 3 \cdot (x+y)\, dydx = \int_0^{0.3} 3 \cdot \left(xy + \frac{y^2}{2} \right) \Bigg|_{0.5}^{1-x} dx =$$
$$= \int_0^{0.3} \left(1.125 - 1.5x - 1.5x^2 \right) dx = \left(\frac{9x}{8} - \frac{3x^2}{4} - \frac{x^3}{2} \right) \Bigg|_0^{0.3} =$$
$$= 0.2565.$$

Let us now check the second case: one at a time.

One at a time

Typical questions:

- Find the marginal distribution of X.
- Find the marginal distribution of Y.

To answer the questions, we need to either express y as a function of x or express x as a function of y.

What we could do wrong: we could possibly allow both x and y to go from 0 to 1, allowing $x + y$ to potentially go higher than 1, breaking the requirement.
Let's see how we could go about solving this:

- Find the marginal distribution of X.

$$f_X(x) = \int_0^{1-x} 3 \cdot (x + y)\, dy = \frac{3}{2} - \frac{3x^2}{2}.$$

- Find the marginal distribution of Y.

$$f_Y(y) = \int_0^{1-y} 3 \cdot (x + y)\, dy = \frac{3}{2} - \frac{3y^2}{2}.$$

The third case involves forgetting that one of the variables exist.

One alone

Typical questions:

- Find the probability that $X \leq 0.3$.
- What is the expectation of Y?

To answer the questions, we simply use the bounds as given!

What we could do wrong: we could possibly try to restrict x or y as a function of the other, when the other no longer exists – as we do not care about it in the setup.

One alone

- Find the probability that $X \leq 0.3$.

$$\int_0^{0.3} f_X(x)dx = \int_0^{0.3} \left(\frac{3}{2} - \frac{3x^2}{2}\right) dx = 0.4365.$$

- What is the expectation of Y?

$$\int_0^{1} y f_Y(y)dy = \int_0^{1} y \left(\frac{3}{2} - \frac{3y^2}{2}\right) dx = 0.375.$$

Extension to more than 2 random variables

Everything we have discussed today can be generalized to more than 2 random variables. [3] More specifically, for a multivariate jointly distributed random variable $(X_1, X_2, \ldots, X_n)$ (hence n random variables), with joint pmf/pdf $f(x_1, x_2, \ldots, x_n)$, we have expectations and variances:

[3] This will prove useful for Lecture 13.

Discrete:

$$E\left[X_i\right] = \sum_x x_i f_{X_i}\left(x_i\right) = \mu_{X_i}$$
$$Var\left[X_i\right] = \sum_x \left(x_i - \mu_{X_i}\right)^2 f_{X_i}\left(x_i\right) = \sigma^2_{X_i}$$

Continuous:

$$E\left[X_i\right] = \int_{-\infty}^{+\infty} x_i f_{X_i}\left(x_i\right) dx_i = \mu_{X_i}$$
$$Var\left[X_i\right] = \int_{-\infty}^{+\infty} \left(x_i - \mu_{X_i}\right)^2 f_{X_i}\left(x_i\right) dx_i = \sigma^2_{X_i}$$

Similarly, for independence:

Definition 4 (Independence) *Random variables $X_1, X_2, \ldots, X_n$ are independent if and only if for all values of $x_1, x_2, \ldots, x_n$, we have that*

$$f_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) \cdots f_{X_n}(x_n).$$

A 4-component machine

Recall the machine (from Lecture 11) that consists of four components, whose lifetimes (in years) are jointly distributed with the following pdf:

$$f_{X_1X_2X_3X_4}(x_1, x_2, x_3, x_4) = 3 \cdot e^{-2x_1}e^{-x_2}e^{-3x_3}e^{-0.5x_4}.$$

Are the random variables from the pdf in this example independent?

First, we calculate all 4 marginal pdfs:

1. $f_{X_1}(x_1) = 2e^{-2x_1}$
2. $f_{X_2}(x_2) = e^{-x_2}$
3. $f_{X_3}(x_3) = 3e^{-3x_3}$
4. $f_{X_4}(x_4) = 0.5e^{-0.5x_4}$

Finally, we have that

$$f_{X_1X_2X_3X_4}(x_1, x_2, x_3, x_4) = f_{X_1}(x_1)\, f_{X_2}(x_2)\, f_{X_3}(x_3)\, f_{X_4}(x_4),$$

so they are independent.

We finish today's notes with the following **very important result**: For a series of random variables, we have:

$$\begin{aligned} E\left[\sum_{i=1}^{n} a_i X_i\right] &= \sum_{i=1}^{n} a_i E[X_i] \\ Var\left[\sum_{i=1}^{n} a_i X_i\right] &= \sum_{i=1}^{n} a_i^2 Var[X_i] + \sum_{i=1}^{n}\sum_{j=1:i\neq j}^{n} Cov[X_i, X_j] \\ &= \sum_{i=1}^{n} a_i^2 Var[X_i] + 2 \cdot \sum_{i=1}^{n}\sum_{i<j} Cov[X_i, X_j] \end{aligned}$$

Since when we have independence, $Cov[X_i, X_j] = 0$ for all X_i, X_j, then, for multiple independent random variables, we have: [4]

[4] We had already derived this result! Check Lecture 9.

$$\begin{aligned} E\left[\sum_{i=1}^{n} a_i X_i\right] &= \sum_{i=1}^{n} a_i E[X_i] \\ Var\left[\sum_{i=1}^{n} a_i X_i\right] &= \sum_{i=1}^{n} a_i^2 Var[X_i] \end{aligned}$$