Counting

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Lecture 2

Learning objectives

After this lecture, we will be able to:

- Count how many outcomes satisfy an event.
- Recall the multiplication rule to count.
- Use the multiplication rule to count.
- Differentiate between permutations and combinations.
- Use permutations and combinations to count.
- Differentiate between different types of permutations.
- Interpret probabilities and recall fundamental probability properties.

Motivation: quantifying probabilities

When we discuss **probability**, there are two worldviews:

1. the frequentist view: which states that the probability of an event happening represents a relative frequency of the times the event happens versus all the times the random experiment is conducted ("in the long run").
2. the Bayesian view: which states that probability is a subjective measure of quantifying the likelihood of an event happening (as a "degree of belief").

Definition 1 (Probability) *With every event, we associate a real number called probability to represent the likelihood of that event happening. Probabilities satisfy three main rules* [1]*:*

1. $P(E) \geq 0$*, for any event* E*.*
2. *If an event* E *comprises the whole sample space (in which case, we write that* $E = S$*), then* $P(E) = 1$*.*
3. *If* $E_1, E_2, \ldots, E_m$ *are* m *mutually exclusive events* [2]*, then*

$$P\left(E_1 \cup E_2 \cup \ldots \cup E_m\right) = P\left(E_1\right) + P\left(E_2\right) + \ldots + P\left(E_m\right),$$

   or even more concisely:

$$P\left(\bigcup_{i=1}^{m} E_i\right) = \sum_{i=1}^{m} P(E_i).$$

[1] Also known as the Kolmogorov axioms of probability.

[2] See the previous lecture.

From the definition, we can deduce that all probabilities are in $[0, 1]$, where a probability of 0 implies that an event can never happen, and a probability of 1 implies that an event is certain (will always happen).

Motivation: equally likely outcomes

When the outcomes in a discrete random experiment with sample space S are **equally probable**, we assume that the probability of each outcome happening is $\frac{1}{|S|}$. Hence, our question becomes:

> "how can we count all favorable outcomes and contrast them to all possible outcomes to derive a measure of probability?"

Why would that be useful?

Counting

The multiplication rule

In the previous lecture and worksheet, we fully enumerated all possible outcomes. For example, rolling two dies results in a total of 36 outcomes:

$$S = \{(1,1), (1,2), (1,3), \ldots, (1,6), (2,1), \ldots, (6,6)\}.$$

What happens if I need to find the cardinality of the sample space of rolling 10 dies?

> A new burrito restaurant
>
> In a new burrito place, you are allowed to choose *only one* of two types of tortillas (flour and wheat), *only one* of four types of "meats" (chicken, pork, steak, no meat), and *only one* of two types of beans (refried and black beans). A food critic needs to try one burrito every day until they have tried all possible burritos. How many days will they be visiting the restaurant to do that?

When our outcomes arise from a sequence of k steps, each of them with n_i, $i = 1, \ldots k$ options (i.e., n_1 options in step 1, n_2 options in step 2, and so on), then

the number of possible outcomes is $n_1 \cdot n_2 \cdot \ldots \cdot n_k$.

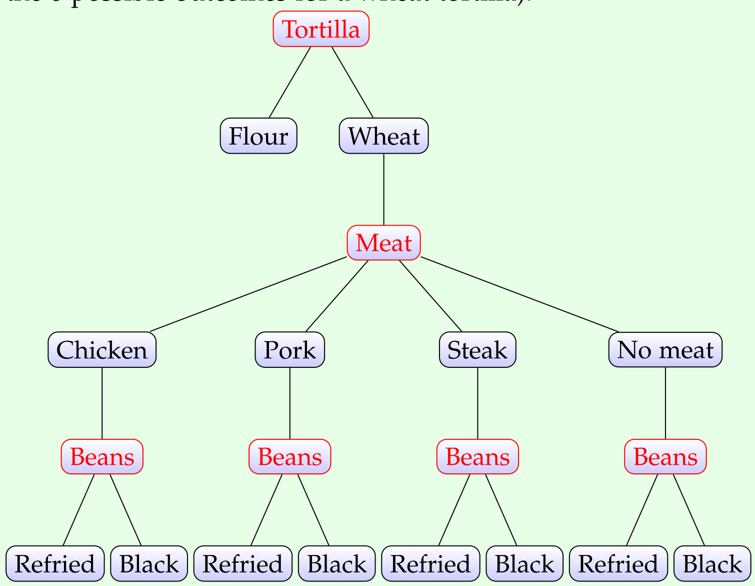
Two key observations:

1. at each step i, we can have to pick exactly one of the n_i options.

2. the order does not matter.

A new burrito restaurant

Hence, in our burrito place example, we have 3 options (tortilla type, meat type, bean type), leading to a total of $2 \cdot 4 \cdot 2 = 16$ combinations (in the figure below, we show the 8 possible outcomes for a wheat tortilla).



In Greece, a vehicle is required to have a license plate with 3 letters (from the Greek alphabet!) and 4 numbers (integer numbers between 0 and 9). How many plates can there be, seeing as the Greek alphabet has 24 letters?

Permutations

A permutation is an **ordered** sequence of elements selected from some set. For example, consider the sample space $S = \{1, 2, 3\}$. All permutations are:

- $\{1, 2, 3\}$
- $\{1, 3, 2\}$
- $\{2, 1, 3\}$
- $\{2, 3, 1\}$
- $\{3, 1, 2\}$
- $\{3, 2, 1\}$

The number of permutations for a sample space with n possible outcomes is [3]

$$P_n = n!$$

[3] $n!$ is defined for any integer number as $n \cdot (n-1) \cdot (n-2) \cdot 1$. $n!$ is read as "n factorial". By definition, we say that $0! = 1$.

A random draw

5 people have been named the winners of a competition. There are 5 different books that will be given to them. How many different outcomes (assignments of winners to books) do we have?

There are $P_5 = 5! = 120$ possible outcomes (assignments of winners to books).

We can also opt to select only $r < n$ from the available elements in the set. For example, consider the set $S = \{1, 2, 3, 4\}$. The permutations of $r = 2$ elements from that set are:

- $\{1, 2\}$
- $\{1, 3\}$
- $\{1, 4\}$
- $\{2, 1\}$
- $\{2, 3\}$
- $\{2, 4\}$
- $\{3, 1\}$
- $\{3, 2\}$
- $\{3, 4\}$
- $\{4, 1\}$
- $\{4, 2\}$
- $\{4, 3\}$

The number of permutations of r outcomes from a total of n outcomes is:

$$P_{n,r} = n \cdot (n-1) \cdot \ldots \cdot (n-r) = \frac{n!}{(n-r)!}$$

A random draw (cont'd)

A total of 100 people are participating in a draw. 5 of the participants will be named winners and get one of 5 different books. How many different outcomes (assignments of winners to books) do we have now?
There are $P_{100,5} = \frac{100!}{(95)!} = 9034502400$ possible outcomes (assignments of winners to books).

Another type of permutation arises when some of the outcomes are the same (for example, two entries in a competition belonging to the same person). In that case, there are fewer **distinguishable permutations**. Formally, assume that:

- we have k different types of outcomes;
- n_1 outcomes of type 1, n_2 outcomes of type 2, $\ldots$, n_k ourcomes of type k;
- such that $n_1 + n_2 + \ldots + n_k = n$.

How many different permutations can we obtain? As an example, assume we are given the following 5 letters in Scrabble: $2 \times E$, $2 \times S$, $1 \times T$. Some of the possible possible 5-letter "words" we can create are:

- EESST
- EESTS
- EETSS
- ETESS
- TEESS
- SEEST
- SEETS
- SETES
- STEES
- TSEES
- ESEST
- ...

> Why is this setup different than the previous permutations we discussed?

If we have k types of elements with n_i objects of type i ($i = 1, \ldots, k$) such that $\sum_{i=1}^{k} n_i = n$, then the number of distinguishable permutations is:

$$\frac{n!}{n_1! \cdot n_2! \cdot \ldots n_k!}.$$

A game of Scrabble

How many different 7-letter words (maybe nonsensical) can we create in a game of Scrabble, where we have $2 \times A, 1 \times B, 2 \times S, 1 \times T, 1 \times X$?
There are 5 different letters with $n_1 = 2, n_2 = 1, n_3 = 2, n_4 = 1, n_5 = 1$. Hence, the total number of distinguishable, 7-letter words we can create is:

$$\frac{7!}{2! \cdot 1! \cdot 2! \cdot 1! \cdot 1!} = 1260 \text{ words.}$$

Combinations

In all of our discussion so far, *order matters*. Often, though, we do not care about it. For example consider the cases of:

- creating a group of 4 people for a class project.
- checking the numbers on ten dies.
- picking the winning numbers in a lottery.

We define a **combination** as an unordered subset of $r < n$ outcomes selected from a sample spance with n outcomes. The number of all possible combinations is calculated by [4]:

$$C_{n,r} = \binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}$$

[4] $\binom{n}{r}$ is also read as "n choose r".

Permutation or combination?

- Choosing 5 students out of 80 candidates to participate in a group?
- Choosing 5 students out of 80 for 5 specific and different positions in a group?
- Locating 10 different facilities in 10 cities in the USA?
- Locating 2 different headquarter facilities from 50 candidate cities in the USA?

Distinguishing between permutations and combinations

You have to select between 10 students for 3 positions. You are allowed pick the same student for all three positions, if you'd like.

- How many possible outcomes are there if the 3 positions are different?

  We need to use the multiplication rule $10 \cdot 10 \cdot 10 = 1000$ possible outcomes.

- What if you are not allowed to select the same student more than once? Assume again the 3 positions are different.

  We need to use a permutation (10 students for 3 different positions): $P_{10,3} = 720$ potential outcomes.

- What if all positions are actually for the same type of work?

  We now have a combination (10 students for 3 positions): $C_{10,3} = 120$ outcomes.

From counting to calculating probability

As mentioned in the Motivation section, **probability** is a real number between 0 and 1 that quantifies how likely an outcome (event) is. Adopting the frequentist view of probabilities [5] we could possibly count the number of outcomes that are favorable and divide by the total number of possible outcomes and thus estimate probability.

[5] The frequentist view states that the probability of an outcome is the relative frequency with which that outcome appears over all possible outcomes (see the Motivation section).

Quality control

A package is set to leave a factory and be sent to a retailer. The package contains 100 items. We already know that exactly 3 of the 100 items are defective. The quality control team over at the retailer works as follows: they select a sample of 6 items from the 100, and check them. If there are 0 defective items in the selected sample of 6, they accept the package and sell its contents; otherwise, they send the package back. What is the probability that the quality control rejects the package and sends it back?

To answer this question, we decompose the problem into its components. We need to know:

1. how many ways are there to select 6 items from the 100?
2. how many ways are there to have 1, 2, or 3 defective items in the 6 selected?

Let us begin by addressing the first question.

Quality control: How many ways are there to select 6 items from the 100 in the package?

Step 1: How many ways are there to select 6 items from the 100 in the package? This is a *combination* and we get:

$$C_{100,6} = \binom{100}{6} = \frac{100!}{6! \cdot 94!} = 1192052400 \text{ ways.}$$

For the second question, we need to think slightly differently. Let x be the number of defective items and $6 - x$ the number of non-defective items in the sample of 6. Then:

Quality control: How many ways are there to have 1, 2, or 3 items in the sample of 6 selected?

Step 2: How many ways are there to have 1, 2, or 3 defective items from the 3 available in the selected sample of 6? This is another *combination*, albeit requiring more calculations.

- **Step 2a:** How to select $x = 1$ defective items in the sample?

  We would need to pick 1 out of the 3 defective and 5 out of the 97 non-defective!

$$C_{3,1} = \binom{3}{1} = 3.$$
$$C_{97,5} = \binom{97}{5} = 64446024.$$

  We should now use the multiplication rule between the two, as we have to pick one option from the 3 possible options from the first selection and one option from the 64446024 possible ones in the second selection, for a total of

$$3 \cdot 64446024 = 193338072 \text{ ways.}$$

- **Step 2b:** How to select $x = 2$ defective items in the sample?

  Similarly:

$$C_{3,2} = \binom{3}{2} = 3.$$
$$C_{97,4} = \binom{97}{4} = 3464840.$$

  The total is 10394520 ways.

- **Step 2c:** How to select $x = 3$ defective items in the sample?

  Similarly:

$$C_{3,3} = \binom{3}{3} = 1.$$
$$C_{97,3} = \binom{97}{3} = 147440.$$

  The total is 147440 ways.

To finish this example, we need to divide the number of desired outcomes (obtained in Step 2) to the total number of outcomes (obtained in Step 1). Note that we may add the three numbers from Step

2 to calculate the total number of desired outcomes [6].

Quality control: What is the probability?

Step 3: Let $E =$ fail inspection. Then:

$$P(E) = \frac{193338072 + 10394520 + 147440}{1192052400} = \frac{203880032}{1192052400} = 0.171.$$

You pick 3 cards at random from a deck with 52 cards. What is the probability that all 3 are face cards? What is the probability that 2 are face cards?

[6] We observe that the three events (selecting $x = 1$, $x = 2$, or $x = 3$ defective in the sample) are **mutually exclusive** and hence the cardinality of the union of the three events is equal to the summation of the individual cardinalities