Confidence intervals for single population means

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Lecture 20

Learning objectives

After lectures 20–23, we will be able to:

- Build confidence intervals for:
  - unknown means;
  - unknown variances;
  - unknown proportions.
- Build confidence intervals for:
  - the difference between two unknown means;
  - the ratio between two unknown variances;
  - the difference between two unknown proportions.
- Understand the effect of Type I error, or probability α.
- Calculate errors and interval margins.
- Select appropriate sample sizes to keep errors below a limit.

Motivation: Point estimates lie

Assume we are told that a new smartphone has a battery of 24 hours, compared to a battery of 22 hours of the previous iteration. A person upgrades to the new phone to see that their new phone also has the same battery as the older one! Should that surprise them?

Motivation: Elections

During an election, many (*many*) polls are released to capture the momentum of the different political parties and candidates. However, surprises and upsets still happen: does that really mean that polling is off? Or should we start caring about the set of plausible outcomes rather than fixating on a single point estimate?

Quick review

In Part 2 of the class, we discussed parameter estimation. Specifically, we saw that:

- Given a population X...
- distributed with some probability density function $f(x)$...
- but with unknown parameter θ...
- we may estimate θ using an estimator $\hat{\Theta}$...
- and use a sample $X_1, X_2, \ldots, X_n$ from the original population...
- to arrive at a single conclusion: a point estimate $\hat{\theta}$!

We also discussed how "wrong" the estimator is, by calculating its bias, variance, mean square error. **But what about the probability our true parameter θ is smaller than $\hat{\theta}$** (or bigger than, or equal to)?

Motivating question 1

We are interested in estimating the unknown mean battery of a new smartphone (population X includes all new smartphones in circulation). We bought a new phone (a random sample from population X, let us call it X_1) and got that the new battery is equal to 21 hours, smaller than what our previous phone had! We are naturally disappointed, so we start asking questions about this new estimate $\hat{\mu} = 21$. What is the probability that the true battery life of smartphones in X is above 21 hours?

Motivating question 2

We are interested in estimating the unknown proportion of people in support of candidate A. We have interviewed 10 voters (randomly selected) and have found that 7 of them support candidate A. The point estimate then, based on our sample, is that $\hat{p} = 70\%$. What is:

- the probability that the true proportion of voters in support of candidate A is above 70%?
- the probability that the true proportion of voters in support of candidate A is below 70%?
- the probability that the true proportion of voters in support of candidate A is exactly equal to 70%?

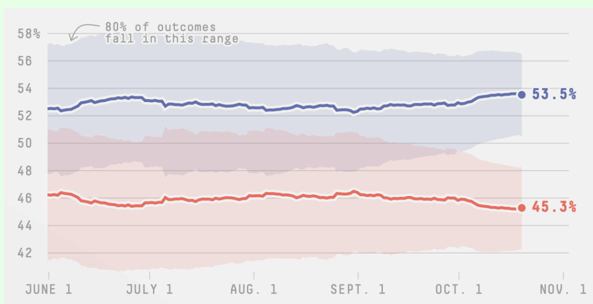
An introduction to confidence intervals

At least one of the previous questions is easy: the probability that the estimate is exactly equal to the true unknown parameter is... zero! This is because for any continuously distributed random variable, $P(X = x) = 0$... So, one thing is for certain: your estimate is not the true parameter value. This is why we introduce **confidence intervals**.

Confidence intervals appear very often in every day life. Some examples include the following.

Election time!

We may want to estimate the proportion of the population preferring one candidate over another. However simply averaging out all the polls is not a solution: we want to reveal all possible scenarios. To do show, we may produce an **interval** of all plausible scenarios and shade them. See the following image taken from 538 (fivethirtyeight.com) on October 20, 2020.



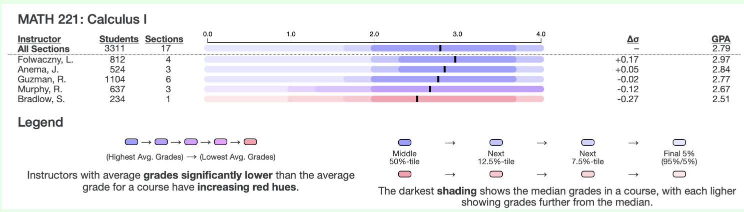
Note the small legend on the upper right corner that "warns" us: 80% of the outcomes fall in the shaded range! So there is still a 20% chance we get something *outside that range*.

Grade disparity

Many students are familiar with a tool, built by by Devin Oliver, Johnny Guo, Joe Tan, Jerry Li, Tina Abraham, Andy (Tianyue) Mao, Kara Landolt, Nathan Cho and Wade Fagen-Ulmschneider (see `https://waf.cs.illinois.edu/discovery/grade_disparity_between_sections_at_uiuc/`) which shows the historical grade distribution for different classes at UIUC. When presenting this information, we do not only want to look at the average GPA that a class has.

Grade disparity (cont'd)

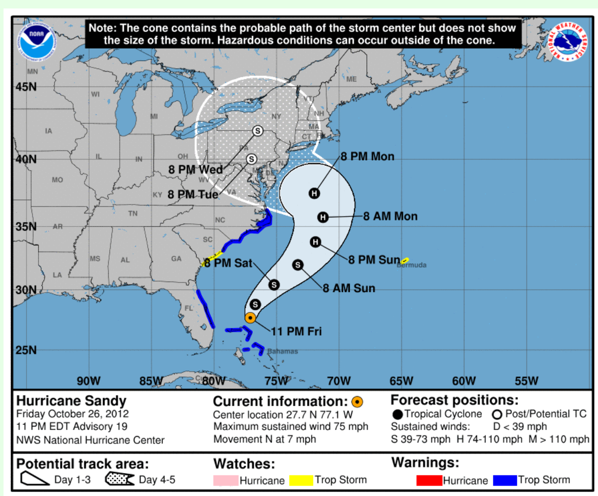
On the contrary, we are also interested in addressing the question: what is the probability a random student ends up with a grade within some range of the expected grade?



And this is clearly presented here, too! Looking at the screenshot, some classes have a very narrow range of values (meaning less deviations from the expected grade), whereas others (look at this last one above!) have a wide array of possible grades.

Cone of uncertainty

For any of us that have lived in a state that gets hit by hurricanes, we have grown used to seeing a map like this one:



This is actually from Hurricane/Superstorm Sandy (end of October 2012). When someone sees this, they may think that this reveals the areas that may be hit, or even that this is the size of the storm. But this is not the case! This "cone of uncertainty" reveals an interval of 60-70% of expected paths based on historical information!

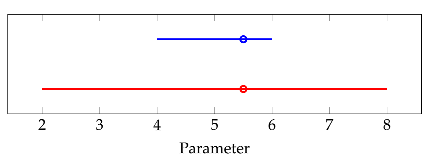
What do all of the above have in common? They are based on this understanding that the true, unknown parameters may be difficult to capture with certainty; so they resort to presenting a series of outcomes (in range form) that reveal a certain percentage of scenarios that can happen. In the election, that was 80% of the scenarios; in the grade disparity case, the first shading includes 50% of the grades historically; in the cone of uncertainty about 60-70% of historical information.

So, let us summarize really quickly before moving to the definition of confidence intervals.

- **Point** estimation: a *single* estimate with our best guess at what the unknown parameter is.
- **Interval** estimation: an interval of values where the unknown parameter is believed to belong in.

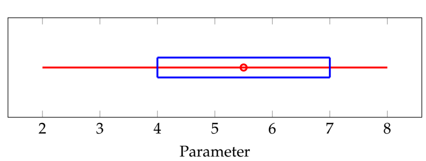
What are the advantages of interval estimation?

1. A point estimation reveals a single point of information about the whereabouts of the unknown parameter, leaving us with no idea of how close the actual parameter is expected to be:



For example, in the above figure the red parameter can be any value between 2 and 8, but our estimate places it at 5.5. On the other hand, the blue one is also estimated at 5.5; however it can only take values within 4 to 6. An interval estimate would reveal more information about these ranges.

2. An interval estimation reveals a margin of error as a measure of accuracy for our parameter.



In this example, we still estimate the red parameter to be 5.5; but now we are also told that we would not be surprised to see it be in any point between 4 and 7.

Now, a **confidence interval**, usually presented in the form of $[L, U]$, contains the most "believable" values for the estimated parameter. Every confidence interval is associated with a **confidence level**, which represents the probability that the true parameter value falls in that interval. In mathematical terms:

$$P(L \leq \theta \leq U) = 1 - \alpha.$$

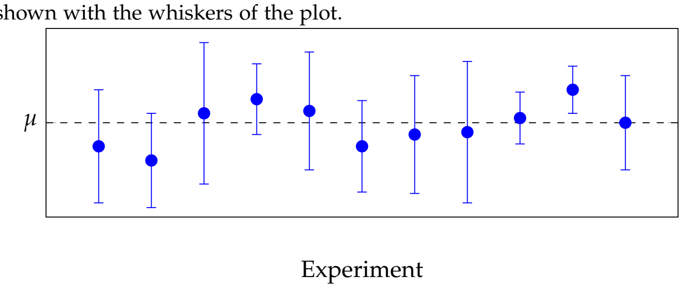
Combining, we write that $[L, U]$ is a $100 \cdot (1 - \alpha)\,\%$ confidence interval for parameter θ.

A couple of notes about confidence intervals and how they came to be. First of all, both L and U are obtained from the random sample we selected. That is, they depend on the sample selected. Secondly, α is an external parameter and we can make it as small or as big as we want to. Smaller α values lead to higher confidence for our interval estimates and vice versa. Typical value for α is 5%, which leads to the creation of 95% confidence intervals. Finally, we may also represent confidence intervals as

$$\boxed{\text{Point estimate} \pm \text{Margin}}$$

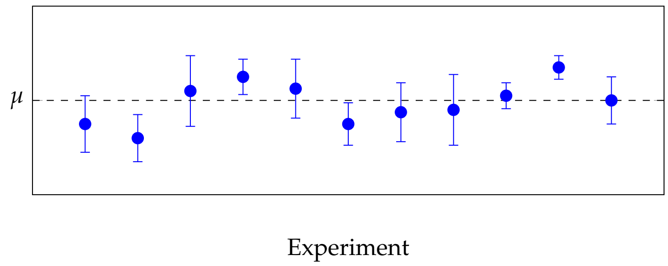
Visually, this is naturally showcased in a plot with the point estimate in the center and the margins on each side as whiskers. As an example we provide Figures 1 and 2.

Figure 1: Here we run 10 experiments and provide the estimate obtained from each of them with a blue dot. We then build a **95% confidence interval** around the estimate; this is shown with the whiskers of the plot.



In both figures we sometimes underestimate and sometimes overestimate the parameter. When we extend our estimate to include a range of "believable" values, we see that the number of experiments that include the true parameter changes sharply. Observe how almost all intervals contain the true mean in the first example; and about half of them do in the second example. As an example take the second experiment: we are underestimating the true value of the parameter through our estimation process. However, in the first figure the 95% confidence interval includes the true parameter μ. In the second figure, it does not.

Figure 2: Here we run 10 experiments and provide the estimate obtained from each of them with a blue dot (note that the estimates are the same as in Figure 1). We then build a **50% confidence interval** around the estimate; this is shown with the whiskers of the plot.



The confidence interval then reveals interesting properties. If we build a 95% confidence interval around an unknown parameter, then this means that:

1. we are 95% certain the parameter is in that range.

2. if we obtain 100 samples, 95 of them will have a parameter in that range.

3. there is a 5% chance we are wrong and the parameter is outside that range (either higher or lower).

Sampling distributions

As a reminder, when picking a sample out of a population, then we say that the sample is distributed with some sampling distribution. For convenience, let us focus on the case of trying to estimate the unknown mean μ of a population X: the population is distributed with some distribution and mean μ (unknown) and variance σ^2 (possibly known).

To estimate μ, we resort to collecting a sample $X_1, X_2, \ldots, X_n$ and calculate the sample average $\overline{X} = \frac{X_1+X_2+\ldots+X_n}{n}$. Recall that the sample average **is a random variable**, distributed with some sampling distribution with:

- expectation $E\left[\overline{X}\right] = \mu$.

- variance $Var\left[\overline{X}\right] = \frac{\sigma^2}{n}$.

Let's distinguish between two cases:

1. X is normally distributed.

2. X is not normally distributed.

If X is normally distributed, the $\overline{X}$ is **also normally distributed** with mean μ and variance σ^2/n. On the other hand, if X is not normally distributed, then $\overline{X}$ is **normally distributed only if the sample size is large enough** (due to the central limit theorem).

Let us assume that one of the above two conditions hold. Then:

- $P(\overline{X} = \mu) = 0$ – $\overline{X}$ is a random variable and the probability it is exactly equal to some other value is zero.
- $P(\overline{X} \geq \mu) = 0.5$ – due to the symmetry of the normal distribution.
- $P(\overline{X} \leq \mu) = 0.5$ – due to the symmetry of the normal distribution.

Recall that in some of our previous worksheets, we had already identified that for a normally distributed random variable, we can calculate the probability of a range of values around the mean as: [1]

[1] See Worksheet 8, Questions 7-8-9.

$$P(\mu - r \leq \overline{X} \leq \mu + r) = 2\Phi\left(\frac{r}{\sigma}\right) - 1$$

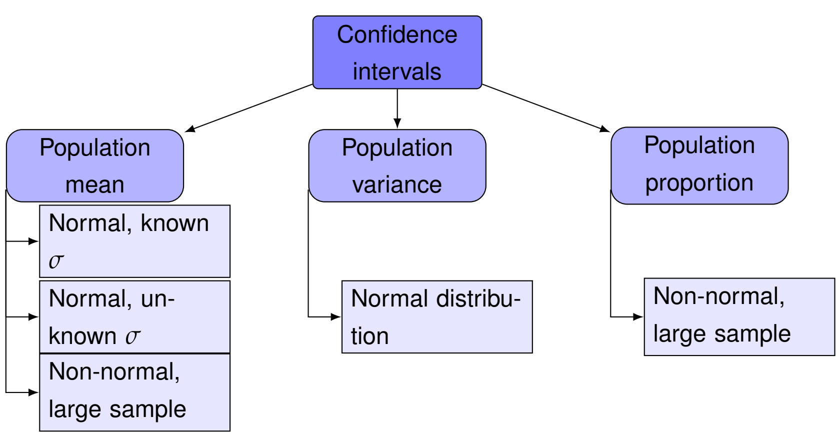
Say, we were looking to build a 95% confidence interval, that would translate to:

$$P(\mu - r \leq \overline{X} \leq \mu + r) = 2\Phi\left(\frac{r}{\sigma}\right) - 1 = 0.95 \implies$$
$$\implies 2\Phi\left(\frac{r}{\sigma}\right) = 1.95 \implies \Phi\left(\frac{r}{\sigma}\right) = 0.975.$$

Hence, $\frac{r}{\sigma}$ has to be the value that leads to 0.975... Let's keep that in the back of our minds for now.

Single population confidence intervals

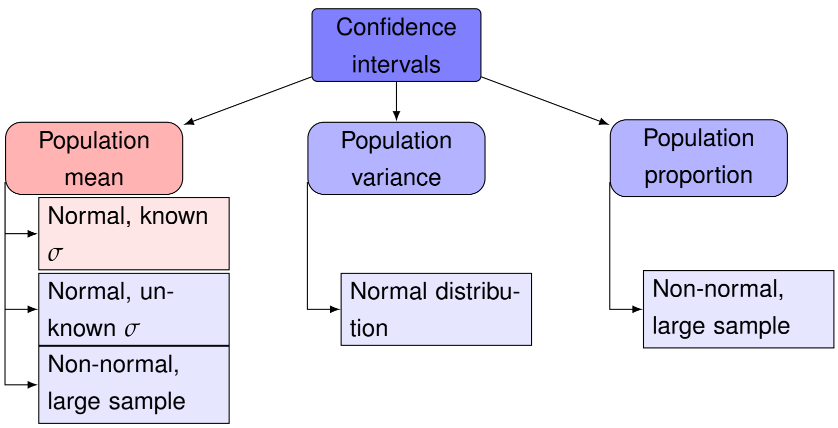
Before we proceed to the next calculations, we provide an overview of where we are headed at:



In all of them we assume the existence of one population with some unknown parameter (the mean, the variance, a proportion). To estimate the unknown parameter, we collect a sample, estimate the unknown parameter based on it and we create an interval around it. We begin with the simplest case: the mean.

Population mean confidence intervals

Assume X is a population with unknown mean. We have collected a sample $X_1, X_2, \ldots, X_n$ to estimate the mean. As we have discussed in previous classes, the sample average $\overline{X}$ is an unbiased estimator for the unknown mean. But what should be the interval around it?



Let us assume that we know X to be normally distributed. And while we are missing the true population mean μ we know its variance σ^2 (or its standard deviation σ).

Recall that we are looking for L, U such that $P(L \leq \overline{X} \leq U) = 1 - \alpha$. As $\overline{X} \sim \mathcal{N}\left(\mu, \sigma^2/n\right)$, we have that $Z = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$. Then, we may write $P(L \leq \overline{X} \leq U) = 1 - \alpha$ as $P(-z_{\alpha/2} \leq Z \leq z_{\alpha/2} = 1 - \alpha$, where $z_{\alpha/2}$ is called the **critical z value** and is found as $P(Z > z_{\alpha/2}) = \frac{\alpha}{2}$. We show what these critical values represent in visual format in Figures 3–6.

Finding critical values

Some common critical values:

- $\alpha = 10\% \implies z_{0.05} = 1.645$ as $\Phi(1.645) = 95\% = 1 - \alpha/2$.
- $\alpha = 5\% \implies z_{0.025} = 1.96$ as $\Phi(1.96) = 97.5\% = 1 - \alpha/2$.
- $\alpha = 1\% \implies z_{0.005} = 2.576$ as $\Phi(2.576) = 99.5\% = 1 - \alpha/2$.

Try it yourselves! What is $z_{\alpha/2}$ for:

- $\alpha = 20\% \implies z_{0.1} =$
- $\alpha = 2\% \implies z_{0.01} =$
- $\alpha = 0.1\% \implies z_{0.0005} =$

Figure 3: $\alpha = 1\%$.

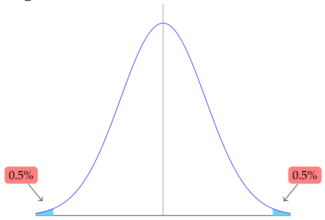


Figure 4: $\alpha = 5\%$.

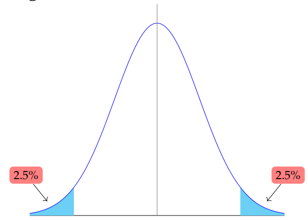


Figure 5: $\alpha = 10\%$.

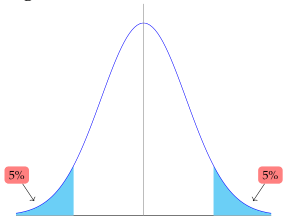
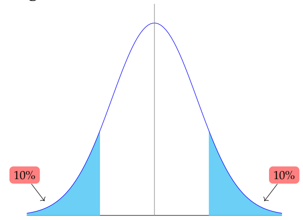


Figure 6: $\alpha = 20\%$.



Based on this discussion, and based on the symmetry of the normal distribution, we have our first confidence interval:

$$\boxed{P(\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) = 1 - \alpha}$$

And consequently, we have a lower bound for our interval at $L = \overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ and an upper bound at $U = \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$.

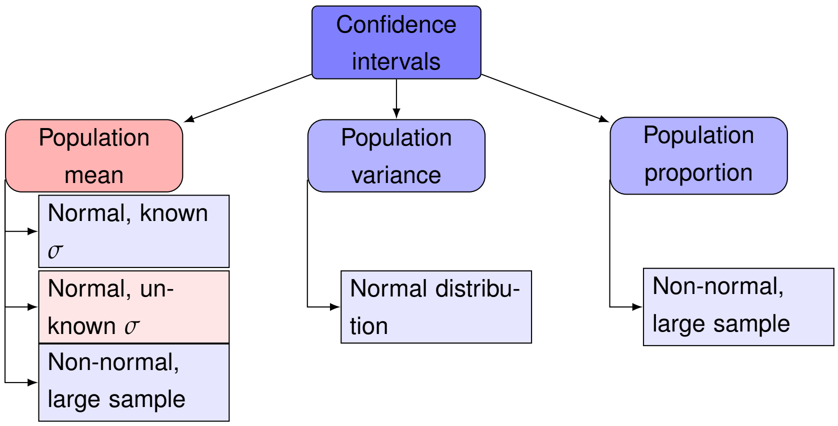
Our first confidence interval

A class at UIUC gives out grades that are normally distributed with known variance equal to 100 (i.e., $\sigma = 10$). Build a 95% confidence interval for the mean of the class grades, assuming that in the previous 8 semesters, the average has been a 77.

First, a 95% confidence interval implies that $\alpha = 0.05$. Hence, we are looking at $z_{\alpha/2} = z_{0.025} = 1.96$. Then our interval will be

$$[L, U] = \left[77 - 1.96 \cdot 10/\sqrt{8}, 77 + 1.96 \cdot 10/\sqrt{8}\right] = [70.07, 83.93].$$

Let us move on to the second part of our discussion about means. What if we know that X is normally distributed but we have no idea what the variance or standard deviation is?

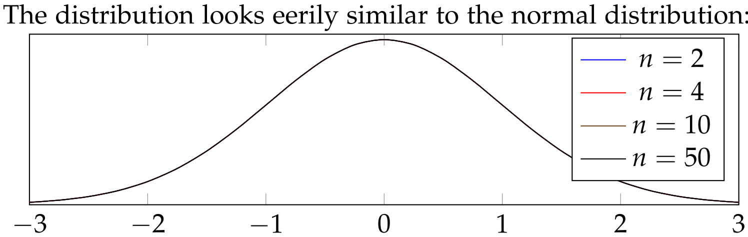


Since we do not know σ, we need to estimate it. And what better way to estimate σ other than using the sample standard deviation s! What is the big deal? Can't we just do everything we did earlier, and simply use s instead of σ?

The short answer is **no**. Unfortunately the statistic $\frac{\overline{X}-\mu}{s/\sqrt{n}}$ which was earlier normally distributed because we knew σ is not any more. Replacing σ with s leads to the statistic to be distributed with the so-called **Student's T distribution**. More specifically, we write that $T = \frac{\overline{X}-\mu}{s/\sqrt{n}}$ is distributed following a Student's T distribution with $n-1$ degrees of freedom.

What kind of name is that? The distribution was introduced by W.S. Gosset, who published his findings under the fake name "Student". This happened because the Guinness brewery (where he was employed at that time) did not allow its employees to publish their findings.

The distribution looks eerily similar to the normal distribution:



It is symmetric, but it has thicker tails. As the degrees of freedom increase, then it starts looking more and more like the actual normal distribution. Observe how for large values of n (say, $n \to \infty$) the z and the t values are identical!

Finally, just like the normal distribution, we may calculate any value we are interested in by looking up a table of values. The table is offered in the last page of the notes for convenience.

Similarly to before then, after replacing z (normal distribution) with t (Student's T distribution) values and replacing σ (known standard deviation) with s (sample standard deviation), we get:

$$\boxed{P(\overline{X} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} \leq \mu \leq \overline{X} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}) = 1 - \alpha}$$

And consequently, we have a lower bound for our interval at $L = \overline{X} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}$ and an upper bound at $U = \overline{X} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}$.

Finding critical values for the T distribution

Some critical values for the T distribution:

- $t_{0.025,15} = 2.131$
- $t_{0.05,10} = 1.812$
- $t_{0.05,25} = 1.708$
- $t_{0.10,5} = 1.476$

STUDENT'S t CRITICAL VALUES

ν	0.4	0.33	0.25	0.2	0.125	0.1	0.05	0.025	0.01	0.005	0.001
1	0.325	0.577	1.000	1.376	2.414	3.078	6.314	12.706	31.821	63.657	318.31
5	0.267	0.457	0.727	0.920	1.301	1.476	2.015	2.571	3.365	4.032	5.893
10	0.260	0.444	0.700	0.879	1.221	1.372	1.812	2.228	2.764	3.169	4.144
15	0.258	0.439	0.691	0.866	1.197	1.341	1.753	2.131	2.602	2.947	3.733
25	0.256	0.436	0.684	0.856	1.178	1.316	1.708	2.060	2.485	2.787	3.450
∞	0.253	0.431	0.674	0.842	1.150	1.282	1.645	1.960	2.326	2.576	3.090

Another confidence interval

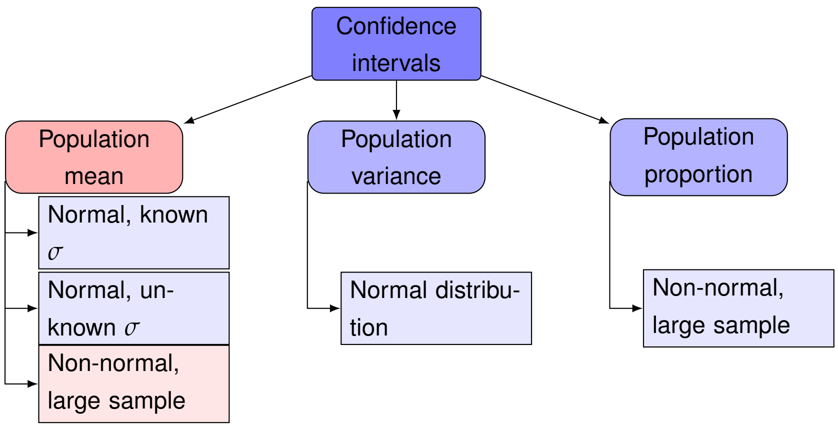
The same class at UIUC gives out grades that are normally distributed with unknown variance; we have observe that over the last 8 semesters the sample variance was equal to 100 (i.e., $s = 10$). Build a 95% confidence interval for the mean of the class grades, assuming that in the previous 8 semesters, the average has been a 77.

We still use $\alpha = 0.05$. Now, though, we are looking at $t_{\alpha/2,7} = t_{0.025,7} = 2.365$. Finally, our interval will be

$$[L, U] = \left[77 - 2.365 \cdot 10/\sqrt{8}, 77 + 2.365 \cdot 10/\sqrt{8}\right] = [68.64, 85.36].$$

Note how the confidence interval has been extended, due to the fact that we do not know what σ is.

For the last case, we will be assuming a general distribution (not necessarily normal) with known or unknown σ: however we also assume the existence of a large enough sample (say, $n \geq 30$).



This case is very similar to the first one. If the sample is big enough, then the **central limit theorem** applies and the average $\overline{X}$ is still normally distributed. If we know σ, we may use it in our calculations; if we do not, then we replace it with the sample standard deviation s. All in all:

$$\boxed{P(\overline{X} - z_{\alpha/2}\tfrac{s}{\sqrt{n}} \leq \mu \leq \overline{X} + z_{\alpha/2}\tfrac{s}{\sqrt{n}}) = 1 - \alpha}$$

Consequently, the confidence interval is given as

$$\left[\overline{X} - z_{\alpha/2}\frac{s}{\sqrt{n}}, \overline{X} + z_{\alpha/2}\frac{s}{\sqrt{n}}\right].$$

Extensions In all of our previous discussion, we assumed **two-sided** confidence intervals. However, in some instance we only care about the one side. Consider the following examples:

- contamination levels (only interested if they are too high);
- grades (only interested if they are too low);
- cholesterol levels (only interested if they are too high);
- and others.

In these cases, we want to build **one-sided** confidence intervals that look like

$$[L, +\infty) \quad \text{or} \quad (-\infty, U].$$

What is the repercussion of having one side rather than two sides? Recall that we choose a level of confidence $1 - \alpha$. With two sides,

Figure 7: $\alpha = 5\%$ (upper bound only).

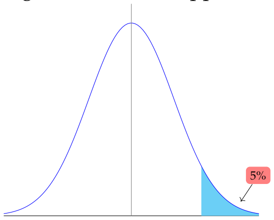


Figure 8: $\alpha = 5\%$ (lower bound only).

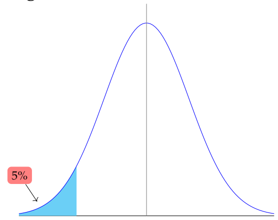


Figure 9: $\alpha = 10\%$ (upper bound only).

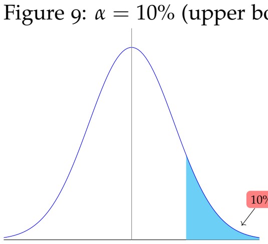
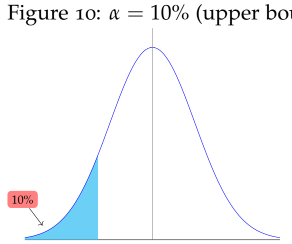


Figure 10: $\alpha = 10\%$ (upper bound only).



this was divided evenly on both sides! Now, all of α gets on one side. Visually, we have the situation of Figures 7–10.

This, in essence, is all that changes: instead of $z_{\alpha/2}$ or $t_{\alpha/2,n-1}$ use z_α or $t_{\alpha,n-1}$ and only calculate a lower or an upper bound as needed.

Cholesterol testing

When testing for cholesterol the sample that a patient has given is divided into 5 parts, each of which is tested individually: assume that each individual test has known $\sigma = 8$. The average measurement was 194; the upper limit out of which the patient may need to start being careful is 200. What is the two-sided 95% confidence interval? What is the one-sided upper 95% confidence interval?
We know the standard deviation, so we are in the first of the three cases we discussed. Hence, the average measurement $\overline{X}$ is normally distributed.

- Two-sided: $z_{\alpha/2} = z_{0.025} = 1.96$.

$$[L, U] = \left[194 - 1.96 \cdot 8/\sqrt{5}, 194 + 1.96 \cdot 8/\sqrt{5}\right] = [186.99, 201.01] .$$

- One-sided: $z_\alpha = z_{0.05} = 1.645$.

$$(-\infty, U] = \left(-\infty, 194 + 1.645 \cdot 8/\sqrt{5}\right] = (-\infty, 199.89] .$$

Let's see the implication of this. The doctor cannot be 95% certain that your cholesterol level is below 200 units if they want to give you a two-sided interval. They can be 95% certain though if they do a one-sided interval!

Another very interesting topic has to do with the question: **how big a sample guarantees me a small error**? The question has two components to address. What do we define as error? And what do we define as small error?

The **estimation error** is the absolute difference between the measured and the true value:

$$E = \left|\overline{X} - \mu\right| \leq z_{\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

The **precision error** is the width of the confidence interval:

$$2z_{\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

Increasing n will have a positive effect on both errors: they go down when we collect more samples. But, of course, the natural question is *how many samples are enough*? Enough for what? This leads us to the following result.

For sample size $n = \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2$, the estimation error is at most E.

For a one-sided interval, the estimation error is $z_\alpha \frac{\sigma}{\sqrt{n}}$ and we need sample size $n = \left(\frac{z_\alpha \sigma}{E}\right)^2$ for the error to be at most E.

As we are discussing the number of samples to obtain, we always round up the number we get if it is fractional.

Cholesterol testing

The doctor from earlier would like to get as many samples as necessary to be sure that your true cholesterol levels are within a estimation error of 7 units. How many samples should they take for a 95% two-sided and a 95% upper-sided (one-sided) confidence interval? Recall that we know $\sigma = 8$.

- two-sided: $z_{\alpha/2} = z_{0.025} = 1.96$:

$$n = \left(\frac{1.96 \cdot 8}{7}\right)^2 = 5.0176 \to 6.$$

- one-sided: $z_\alpha = z_{0.05} = 1.645$:

$$n = \left(\frac{1.645 \cdot 8}{7}\right)^2 = 3.5344 \to 4.$$

NORMAL CUMULATIVE DISTRIBUTION FUNCTION ($\Phi(z)$)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

STUDENT'S *t* CRITICAL VALUES

ν	0.4	0.33	0.25	0.2	0.125	0.1	0.05	0.025	0.01	0.005	0.001
1	0.325	0.577	1.000	1.376	2.414	3.078	6.314	12.706	31.821	63.657	318.31
2	0.289	0.500	0.816	1.061	1.604	1.886	2.920	4.303	6.965	9.925	22.327
3	0.277	0.476	0.765	0.978	1.423	1.638	2.353	3.182	4.541	5.841	10.215
4	0.271	0.464	0.741	0.941	1.344	1.533	2.132	2.776	3.747	4.604	7.173
5	0.267	0.457	0.727	0.920	1.301	1.476	2.015	2.571	3.365	4.032	5.893
6	0.265	0.453	0.718	0.906	1.273	1.440	1.943	2.447	3.143	3.707	5.208
7	0.263	0.449	0.711	0.896	1.254	1.415	1.895	2.365	2.998	3.499	4.785
8	0.262	0.447	0.706	0.889	1.240	1.397	1.860	2.306	2.896	3.355	4.501
9	0.261	0.445	0.703	0.883	1.230	1.383	1.833	2.262	2.821	3.250	4.297
10	0.260	0.444	0.700	0.879	1.221	1.372	1.812	2.228	2.764	3.169	4.144
11	0.260	0.443	0.697	0.876	1.214	1.363	1.796	2.201	2.718	3.106	4.025
12	0.259	0.442	0.695	0.873	1.209	1.356	1.782	2.179	2.681	3.055	3.930
13	0.259	0.441	0.694	0.870	1.204	1.350	1.771	2.160	2.650	3.012	3.852
14	0.258	0.440	0.692	0.868	1.200	1.345	1.761	2.145	2.624	2.977	3.787
15	0.258	0.439	0.691	0.866	1.197	1.341	1.753	2.131	2.602	2.947	3.733
16	0.258	0.439	0.690	0.865	1.194	1.337	1.746	2.120	2.583	2.921	3.686
17	0.257	0.438	0.689	0.863	1.191	1.333	1.740	2.110	2.567	2.898	3.646
18	0.257	0.438	0.688	0.862	1.189	1.330	1.734	2.101	2.552	2.878	3.610
19	0.257	0.438	0.688	0.861	1.187	1.328	1.729	2.093	2.539	2.861	3.579
20	0.257	0.437	0.687	0.860	1.185	1.325	1.725	2.086	2.528	2.845	3.552
21	0.257	0.437	0.686	0.859	1.183	1.323	1.721	2.080	2.518	2.831	3.527
22	0.256	0.437	0.686	0.858	1.182	1.321	1.717	2.074	2.508	2.819	3.505
23	0.256	0.436	0.685	0.858	1.180	1.319	1.714	2.069	2.500	2.807	3.485
24	0.256	0.436	0.685	0.857	1.179	1.318	1.711	2.064	2.492	2.797	3.467
25	0.256	0.436	0.684	0.856	1.178	1.316	1.708	2.060	2.485	2.787	3.450
26	0.256	0.436	0.684	0.856	1.177	1.315	1.706	2.056	2.479	2.779	3.435
27	0.256	0.435	0.684	0.855	1.176	1.314	1.703	2.052	2.473	2.771	3.421
28	0.256	0.435	0.683	0.855	1.175	1.313	1.701	2.048	2.467	2.763	3.408
29	0.256	0.435	0.683	0.854	1.174	1.311	1.699	2.045	2.462	2.756	3.396
30	0.256	0.435	0.683	0.854	1.173	1.310	1.697	2.042	2.457	2.750	3.385
40	0.255	0.434	0.681	0.851	1.167	1.303	1.684	2.021	2.423	2.704	3.307
50	0.255	0.433	0.679	0.849	1.164	1.299	1.676	2.009	2.403	2.678	3.261
60	0.254	0.433	0.679	0.848	1.162	1.296	1.671	2.000	2.390	2.660	3.232
∞	0.253	0.431	0.674	0.842	1.150	1.282	1.645	1.960	2.326	2.576	3.090