Confidence intervals for single population variances and proportions

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Lecture 21

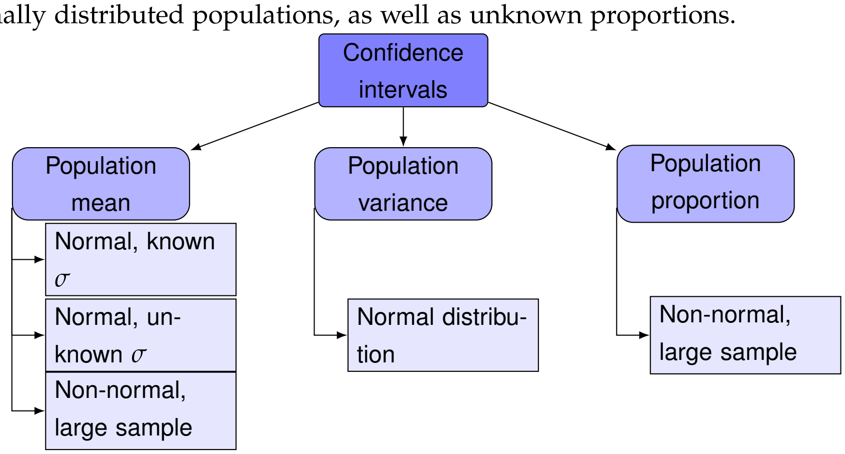
Learning objectives

After lectures 20–23, we will be able to:

- Build confidence intervals for:
  - unknown means;
  - unknown variances;
  - unknown proportions.
- Build confidence intervals for:
  - the difference between two unknown means;
  - the ratio between two unknown variances;
  - the difference between two unknown proportions.
- Understand the effect of Type I error, or probability α.
- Calculate errors and interval margins.
- Select appropriate sample sizes to keep errors below a limit.

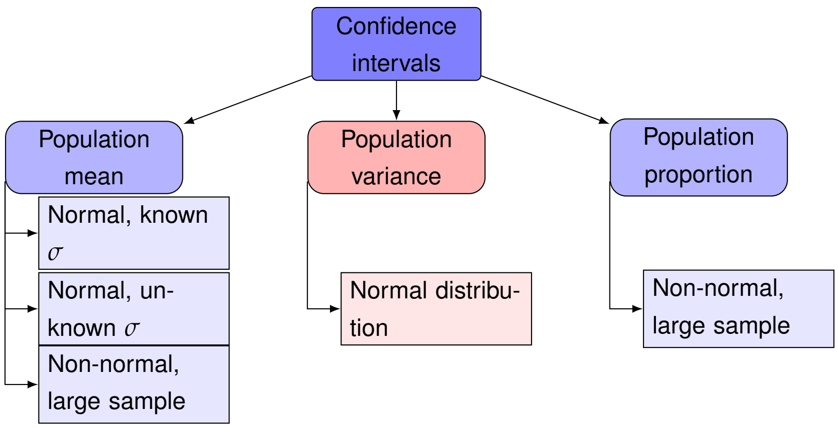
Single population confidence intervals

Continuing from last lecture, we are still building confidence intervals for a single population. In this lecture, though, we will talk about creating confidence intervals for unknown variances of normally distributed populations, as well as unknown proportions.



Population variance confidence intervals

Assume X is a normally distributed population with unknown variance. We have collected a sample $X_1, X_2, \ldots, X_n$ to estimate the variance. As we have discussed in previous classes, the sample variance s^2 is an unbiased estimator for the unknown variance. But what should be the interval around it? This is our focus:



Once again, we are looking for L, U such that $P(L \leq s^2 \leq U) = 1 - \alpha$. However, we first need to discuss what s^2 is distributed as.

Sampling distribution for σ^2

Recall that we have a good estimator for the population variance σ^2:

- pick a sample $X_1, X_2, \ldots, X_n$.
- estimate the variance by the sample variance: s^2.

We have already proven that $E\left[s^2\right] = \sigma^2$. The question now is: what is the sampling distribution of s^2? It turns out it follows the χ^2 **distribution**. [1] The distribution is formally defined as follows:

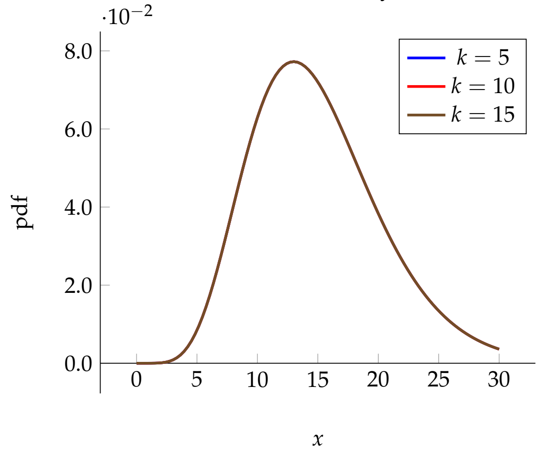
> Let $X_1, X_2, \ldots, X_n$ be a sample from a normally distributed population with $\mathcal{N}\left(\mu, \sigma^2\right)$. Then, the random variable
>
> $$X^2 = \frac{(n-1)\,s^2}{\sigma^2}$$
>
> is distributed with a χ^2-distribution with $n - 1$ degrees of freedom.

It is the sum of the squares of $n-1$ normally distributed random variables. For a visual representation see Figure 1.

Very similarly to our previous operations for other confidence intervals, we again focus on identifying critical values for the χ^2-

[1] Pronounced "Chi-Squared".

Figure 1: Here we present the χ^2 distribution for three different degrees of freedom equal to $k = 5, 10, 15$. Note how the distribution is **not symmetric**.



distribution, that is values such that:

$$P(X^2 \geq \chi^2_{\alpha,k}) = \alpha.$$

Luckily, we again may use a table containing these values, referred to as (you guessed it) a χ^2-table.

Practice with the χ^2 distribution

For example, let us practice with some values:

- $\chi^2_{0.05,5} = 11.07$
- $\chi^2_{0.1,5} = 9.236$
- $\chi^2_{0.9,20} = 12.443$
- $\chi^2_{0.95,55} = 38.958$

Here are some values taken from the tables in the last two pages. These should help with finding the above critical values. Again, we look at the rows for the degrees of freedom, and at the columns for the percentages.

ν	99%	97.5%	95%	90%	10%	5%	2.5%	1%
1	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635
5	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086
10	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209
20	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566
55	33.570	36.398	38.958	42.060	68.796	73.311	77.380	82.292

Once more, assume we have a sample $X_1, X_2, \ldots, X_n$. Then:

$$X^2 = \frac{(n-1)\,s^2}{\sigma^2} \sim \chi^2_{n-1}$$

and hence:

$$P\left(\chi^2_{1-\alpha/2,n-1} \leq X^2 \leq \chi^2_{\alpha/2,n-1}\right) = 1 - \alpha.$$

By converting back to the σ^2 space, we get:

$$P\left(\frac{(n-1)\,s^2}{\chi^2_{\alpha/2,n-1}} \leq \sigma^2 \leq \frac{(n-1)\,s^2}{\chi^2_{1-\alpha/2,n-1}}\right),$$

where the two bounds are (in $[L, U]$ form):

$$L = \frac{(n-1)\,s^2}{\chi^2_{\alpha/2,n-1}}$$

$$U = \frac{(n-1)\,s^2}{\chi^2_{1-\alpha/2,n-1}}$$

A couple of notes of caution for when building a variance confidence interval:

1. There are no *actual* squares involved! You do not "square" the value: this is simply the name of the distribution!
2. Notice that the critical values are not symmetric: in the normal and the t distribution, the values are symmetric.
   - For the lower bound, use $\chi^2_{\alpha/2,n-1}$;
   - For the upper bound, use $\chi^2_{1-\alpha/2,n-1}$.
3. Because of the lack of symmetry in the critical values, there is no symmetry in the bounds.
   - On top of that, you are dividing the estimator by a value (rather than adding it and subtracting it to the estimator, which was the case earlier).

Our first variance confidence interval

An engineer is concerned about soil contamination, which is assumed to be normally distributed. They pick 15 soil samples and measure the contaminant levels finding that $\overline{X} = 13.7$ ppm and $s = 3.15$ ppm. You may assume that the soil contamination level has unknown mean and variance. What is:

1. a 95% confidence interval for μ?
2. a 95% confidence interval for σ^2?

A mean confidence interval first

Wait! The first part is for a mean confidence interval. Let us do a quick activity then to find it. We have:

1. normally distributed population;

2. unknown variance.

Hence, we need values from the t-table. More specifically, we need:

- $t_{0.025,14} = 2.145$ to build the mean confidence interval.

This leads to an interval that:

$$\mu \in \left[13.7 - 2.145 \cdot \frac{3.15}{\sqrt{15}}, 13.7 + 2.145 \cdot \frac{3.15}{\sqrt{15}}\right] = [11.96, 15.44] .$$

And a variance confidence interval next

For the variance confidence interval, we look at the χ^2 table (look at the last two pages of this set of nodes) to find the two values we need:

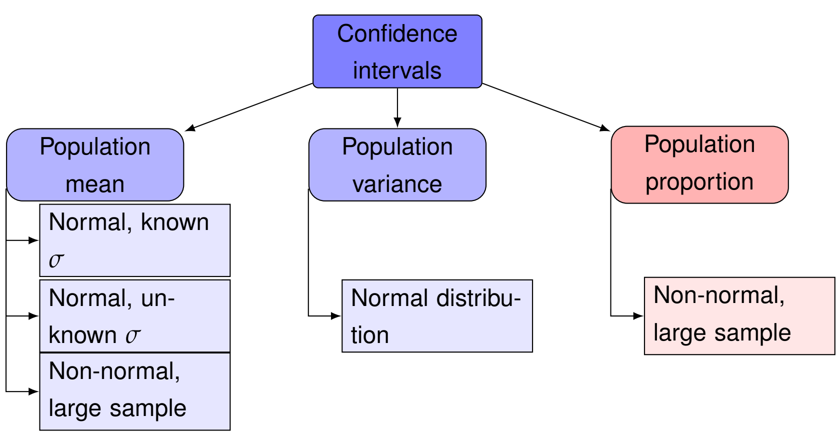
- $\chi^2_{0.025,14} = 26.119, \quad \chi^2_{0.975,14} = 5.629.$

The interval then is found as:

$$\sigma^2 \in \left[\frac{14 \cdot 3.15^2}{26.119}, \frac{14 \cdot 3.15^2}{5.629}\right] = [5.32, 24.68]$$

Note how it is not at all symmetric!

Population proportion confidence intervals



Let us see the last case now. We begin with a motivational example.

Policy making

Assume we are deciding for a new law, and want to make sure that the population of a city (estimated at 100,000) supports it. Moreover, assume that support means 50% or more people like the law.

What can we do?

- Ask a random set of n people whether they support the law.
- Count how many support the law. Let them be X.
- Estimate $\hat{p} = \frac{X}{n}$.

Suppose $\hat{p} = 0.6$ after asking $n = 30$ people.
Should we enact the law? *Are we 95% sure the majority supports it?*

In the previous example, we have that $X \sim \text{binomial}(n, p)$. When n is big enough, then X is approximated by a normal distribution with mean np and variance $np(1-p)$. [2] Let us state this more formally.

[2] Why is that?

Definition 1 (Normal approximation to the binomial distribution) *Assume that X is binomially distributed with parameters n, p. Further assume that $\underline{np > 5}$ and $\underline{n(1-p) > 5}$. Then, X can be written as a normally distributed random variable $\mathcal{N}(np, np(1-p))$.*

Because of that, the statistic $Z = \frac{X - np}{\sqrt{np(1-p)}}$ follows the standard normal distribution (i.e., $\mathcal{N}(0,1)$). Note how we can rewrite Z as follows:

$$Z = \frac{X - np}{\sqrt{np(1-p)}} = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \sim \mathcal{N}(0,1).$$

Now, let us derive the confidence intervals. Let $\hat{p}$ be the proportion of observations that are of interest (for example, the number of people who agree with a statement versus the total number n of people asked). Then:

$$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq p \leq \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Policy making

We asked 30 people and 18 said they support the law. What is the 95%-confidence interval for the true proportion supporting the law in the city?

$$0.6 - 1.96 \cdot \sqrt{\frac{0.6 \cdot 0.4}{30}} \leq p \leq 0.6 + 1.96 \cdot \sqrt{\frac{0.6 \cdot 0.4}{30}} \implies$$
$$0.4247 \leq p \leq 0.7753.$$

Bounding the error

The **estimation error** for our point estimate $\hat{p}$ is

$$E = |\hat{p} - p|.$$

Assume we are asked to calculate a $100 \cdot (1-\alpha)\%$ confidence interval. Then, its error is bounded above by:

$$E \leq z_{\alpha/2}\sqrt{p(1-p)/n}.$$

Expectedly, as n increases, the error bound goes down. But the real question is: **how big should n be for the error to be at a pre-specified level?** We may calculate this as:

$$n \geq \left(\frac{z_{\alpha/2}}{E}\right)^2 p(1-p).$$

However... the true proportion p is unknown – but we can show that $p(1-p) \leq 0.25$ [3]. Hence, we use just that to finally get that:

$$\boxed{n \geq 0.25\left(\frac{z_{\alpha/2}}{E}\right)^2.}$$

[3] This is the maximum value for $p \cdot (1-p)$ for any value of p $[0,1]$.

Policy making

In the previous example, we want to have a 95%-confidence interval with an error of at most $E = 5\%$. How many people should we ask?

95%-confidence level $\implies z_{0.025} = 1.96$. Hence, we get:

$$n \geq 0.25 \cdot \left(\frac{1.96}{0.05}\right)^2 = 384.16 \implies n = 385.$$

We should ask at least 385 people.

Observe that the number does **not** depend on the specific population, but *only* on the confidence level and the pre-specified error.

ν	99.9%	99.5%	99.0%	97.5%	95.0%	90.0%	87.5%	80.0%	75.0%	66.7%	50.0%
1	0.000	0.000	0.000	0.001	0.004	0.016	0.025	0.064	0.102	0.186	0.455
2	0.002	0.010	0.020	0.051	0.103	0.211	0.267	0.446	0.575	0.811	1.386
3	0.024	0.072	0.115	0.216	0.352	0.584	0.692	1.005	1.213	1.568	2.366
4	0.091	0.207	0.297	0.484	0.711	1.064	1.219	1.649	1.923	2.378	3.357
5	0.210	0.412	0.554	0.831	1.145	1.610	1.808	2.343	2.675	3.216	4.351
6	0.381	0.676	0.872	1.237	1.635	2.204	2.441	3.070	3.455	4.074	5.348
7	0.598	0.989	1.239	1.690	2.167	2.833	3.106	3.822	4.255	4.945	6.346
8	0.857	1.344	1.646	2.180	2.733	3.490	3.797	4.594	5.071	5.826	7.344
9	1.152	1.735	2.088	2.700	3.325	4.168	4.507	5.380	5.899	6.716	8.343
10	1.479	2.156	2.558	3.247	3.940	4.865	5.234	6.179	6.737	7.612	9.342
11	1.834	2.603	3.053	3.816	4.575	5.578	5.975	6.989	7.584	8.514	10.341
12	2.214	3.074	3.571	4.404	5.226	6.304	6.729	7.807	8.438	9.420	11.340
13	2.617	3.565	4.107	5.009	5.892	7.042	7.493	8.634	9.299	10.331	12.340
14	3.041	4.075	4.660	5.629	6.571	7.790	8.266	9.467	10.165	11.245	13.339
15	3.483	4.601	5.229	6.262	7.261	8.547	9.048	10.307	11.037	12.163	14.339
16	3.942	5.142	5.812	6.908	7.962	9.312	9.837	11.152	11.912	13.083	15.338
17	4.416	5.697	6.408	7.564	8.672	10.085	10.633	12.002	12.792	14.006	16.338
18	4.905	6.265	7.015	8.231	9.390	10.865	11.435	12.857	13.675	14.931	17.338
19	5.407	6.844	7.633	8.907	10.117	11.651	12.242	13.716	14.562	15.859	18.338
20	5.921	7.434	8.260	9.591	10.851	12.443	13.055	14.578	15.452	16.788	19.337
21	6.447	8.034	8.897	10.283	11.591	13.240	13.873	15.445	16.344	17.720	20.337
22	6.983	8.643	9.542	10.982	12.338	14.041	14.695	16.314	17.240	18.653	21.337
23	7.529	9.260	10.196	11.689	13.091	14.848	15.521	17.187	18.137	19.587	22.337
24	8.085	9.886	10.856	12.401	13.848	15.659	16.351	18.062	19.037	20.523	23.337
25	8.649	10.520	11.524	13.120	14.611	16.473	17.184	18.940	19.939	21.461	24.337
26	9.222	11.160	12.198	13.844	15.379	17.292	18.021	19.820	20.843	22.399	25.336
27	9.803	11.808	12.879	14.573	16.151	18.114	18.861	20.703	21.749	23.339	26.336
28	10.391	12.461	13.565	15.308	16.928	18.939	19.704	21.588	22.657	24.280	27.336
29	10.986	13.121	14.256	16.047	17.708	19.768	20.550	22.475	23.567	25.222	28.336
30	11.588	13.787	14.953	16.791	18.493	20.599	21.399	23.364	24.478	26.165	29.336
35	14.688	17.192	18.509	20.569	22.465	24.797	25.678	27.836	29.054	30.894	34.336
40	17.916	20.707	22.164	24.433	26.509	29.051	30.008	32.345	33.660	35.643	39.335
45	21.251	24.311	25.901	28.366	30.612	33.350	34.379	36.884	38.291	40.407	44.335
50	24.674	27.991	29.707	32.357	34.764	37.689	38.785	41.449	42.942	45.184	49.335
55	28.173	31.735	33.570	36.398	38.958	42.060	43.220	46.036	47.610	49.972	54.335
60	31.738	35.534	37.485	40.482	43.188	46.459	47.680	50.641	52.294	54.770	59.335

ν	40.0%	33.3%	25.0%	20.0%	12.5%	10.0%	5.0%	2.5%	1.0%	0.5%	0.1%
1	0.708	0.936	1.323	1.642	2.354	2.706	3.841	5.024	6.635	7.879	10.828
2	1.833	2.197	2.773	3.219	4.159	4.605	5.991	7.378	9.210	10.597	13.816
3	2.946	3.405	4.108	4.642	5.739	6.251	7.815	9.348	11.345	12.838	16.266
4	4.045	4.579	5.385	5.989	7.214	7.779	9.488	11.143	13.277	14.860	18.467
5	5.132	5.730	6.626	7.289	8.625	9.236	11.070	12.833	15.086	16.750	20.515
6	6.211	6.867	7.841	8.558	9.992	10.645	12.592	14.449	16.812	18.548	22.458
7	7.283	7.992	9.037	9.803	11.326	12.017	14.067	16.013	18.475	20.278	24.322
8	8.351	9.107	10.219	11.030	12.636	13.362	15.507	17.535	20.090	21.955	26.125
9	9.414	10.215	11.389	12.242	13.926	14.684	16.919	19.023	21.666	23.589	27.877
10	10.473	11.317	12.549	13.442	15.198	15.987	18.307	20.483	23.209	25.188	29.588
11	11.530	12.414	13.701	14.631	16.457	17.275	19.675	21.920	24.725	26.757	31.264
12	12.584	13.506	14.845	15.812	17.703	18.549	21.026	23.337	26.217	28.300	32.910
13	13.636	14.595	15.984	16.985	18.939	19.812	22.362	24.736	27.688	29.819	34.528
14	14.685	15.680	17.117	18.151	20.166	21.064	23.685	26.119	29.141	31.319	36.123
15	15.733	16.761	18.245	19.311	21.384	22.307	24.996	27.488	30.578	32.801	37.697
16	16.780	17.840	19.369	20.465	22.595	23.542	26.296	28.845	32.000	34.267	39.252
17	17.824	18.917	20.489	21.615	23.799	24.769	27.587	30.191	33.409	35.718	40.790
18	18.868	19.991	21.605	22.760	24.997	25.989	28.869	31.526	34.805	37.156	42.312
19	19.910	21.063	22.718	23.900	26.189	27.204	30.144	32.852	36.191	38.582	43.820
20	20.951	22.133	23.828	25.038	27.376	28.412	31.410	34.170	37.566	39.997	45.315
21	21.991	23.201	24.935	26.171	28.559	29.615	32.671	35.479	38.932	41.401	46.797
22	23.031	24.268	26.039	27.301	29.737	30.813	33.924	36.781	40.289	42.796	48.268
23	24.069	25.333	27.141	28.429	30.911	32.007	35.172	38.076	41.638	44.181	49.728
24	25.106	26.397	28.241	29.553	32.081	33.196	36.415	39.364	42.980	45.559	51.179
25	26.143	27.459	29.339	30.675	33.247	34.382	37.652	40.646	44.314	46.928	52.620
26	27.179	28.520	30.435	31.795	34.410	35.563	38.885	41.923	45.642	48.290	54.052
27	28.214	29.580	31.528	32.912	35.570	36.741	40.113	43.195	46.963	49.645	55.476
28	29.249	30.639	32.620	34.027	36.727	37.916	41.337	44.461	48.278	50.993	56.892
29	30.283	31.697	33.711	35.139	37.881	39.087	42.557	45.722	49.588	52.336	58.301
30	31.316	32.754	34.800	36.250	39.033	40.256	43.773	46.979	50.892	53.672	59.703
35	36.475	38.024	40.223	41.778	44.753	46.059	49.802	53.203	57.342	60.275	66.619
40	41.622	43.275	45.616	47.269	50.424	51.805	55.758	59.342	63.691	66.766	73.402
45	46.761	48.510	50.985	52.729	56.052	57.505	61.656	65.410	69.957	73.166	80.077
50	51.892	53.733	56.334	58.164	61.647	63.167	67.505	71.420	76.154	79.490	86.661
55	57.016	58.945	61.665	63.577	67.211	68.796	73.311	77.380	82.292	85.749	93.168
60	62.135	64.147	66.981	68.972	72.751	74.397	79.082	83.298	88.379	91.952	99.607