Joint distributions

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Lecture 11

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ISE | Industrial & Enterprise Systems Engineering
GRAINGER COLLEGE OF ENGINEERING

Joint probability distributions

So far in the class we have analyzed **single** random variables or groups of independent random variables.

Definition

Let X and Y be two random variables. The probability distribution that defines their *simultaneous* behavior is referred to as a joint probability distribution.

- How many interviews until you get a job (X) and the state of the economy (Y).
- How many times you repeat something to an automated call system (X) and your cell phone reception (Y).
- The time it takes for a professor to answer a question (X) and the quantity of caffeine in the professor's system (Y).
- The quality of Wi-Fi signal (X) and the number of data packages sent correctly through the router (Y).

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Discrete random variables: the joint pmf

If X and Y are discrete random variables, then (X, Y) is called a **jointly distributed discrete bivariate random variable**.

Definition

The **joint probability mass function** is defined as:

$$f_{XY}(x, y) = P(X = x, Y = y).$$

1. $f_{XY}(x, y) \geq 0, \forall x, y.$
2. $\sum_{x} \sum_{y} f_{XY}(x, y) = 1.$
3. $P((X, Y) \in A) = \sum \sum_{(x,y) \in A} f_{XY}(x, y).$

Easily generalizable to more than two variables: if X_i are discrete random variables for $i = 1, \ldots, n$, then $(X_1, \ldots, X_n)$ is called a **jointly distributed discrete multivariate random variable** with joint pmf:

$$f_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n).$$

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1. The marginal distribution of X:

$$f_X(x) = P(X = x) = \sum_y f_{XY}(x, y)$$

2. The marginal distribution of Y:

$$f_Y(y) = P(Y = y) = \sum_x f_{XY}(x, y)$$

Once again, easily generalizable to more than 2 variables:

- The marginal distribution of X_j:

$$f_{X_j}(x) = P(X_j = x) = \sum_{x_1} \cdots \sum_{x_{j-1}} \sum_{x_{j+1}} \cdots \sum_{x_n} f_{X_1 X_2 \dots X_n}(x_1, x_2, \dots, x_n)$$

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Discrete random variables: the conditional pmf

Finally, the **conditional probability mass function** of a random variable *given* a value for the other random variable can be found as:

1. The conditional distribution of X, given $Y = y$:

$$f_{X|Y=y}(x) = f_{X|y}(x) = \frac{f_{XY}(x,y)}{f_Y(y)}.$$

2. The conditional distribution of Y, given $X = x$:

$$f_{Y|X=x}(y) = f_{Y|x}(y) = \frac{f_{XY}(x,y)}{f_X(x)}.$$

Of course, we need $f_X(x), f_Y(y) > 0$.

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Discrete random variables: an example

Example

Two discrete random variables X and Y have a joint distribution of $f_{XY}(x, y) = \frac{x+y+1}{c}$, *for x and y equal to 0, 1, or 2.*

1. *What should c be?*
2. *What is* $P(X \leq 1, Y = 1)$*?*
3. *What is* $P(X \leq 1 | Y = 1)$*?*

Answer: We have:

$$\sum_x \sum_y f_{XY}(x, y) = 1 \implies \sum_{x=0}^{2} \sum_{y=0}^{2} \frac{x+y+1}{c} = 1 \implies$$

$$\frac{1}{c} + \frac{2}{c} + \frac{3}{c} + \frac{2}{c} + \frac{3}{c} + \frac{4}{c} + \frac{3}{c} + \frac{4}{c} + \frac{5}{c} = 1 \implies c = 27.$$

$$P(X \leq 1, Y = 1) = P(X = 0, Y = 1) + P(X = 1, Y = 1) = \frac{5}{27}.$$

$$P(X \leq 1 | Y = 1) = \frac{P(X \leq 1, Y = 1)}{P(Y = 1)} = \frac{5/27}{9/27} = \frac{5}{9}.$$

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Continuous random variables: the joint pdf

If X and Y are continuous random variables, then (X, Y) is called a **jointly distributed continuous bivariate random variable**.

Definition

The **joint probability distribution function** is denoted by $f_{XY}(x, y)$ and satisfies the following properties:

1. $f_{XY}(x, y) \geq 0, \forall x, y.$
2. $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{XY}(x, y) dxdy = 1.$
3. $P((X, Y) \subset R) = \iint_R f_{XY}(x, y) dxdy.$

Remember: the pdf does **not** reveal probability but relative likelihood.

Once again generalizable to more than two variables: if X_i are continuous random variables for $i = 1, \ldots, n$, then $(X_1, \ldots, X_n)$ is called a **jointly distributed continuous multivariate random variable** with joint pdf:

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The **marginal probability density function** is defined by integrating over one of the random variables. We can obtain:

1. The marginal pdf of X:

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y) dy.$$

2. The marginal pdf of Y:

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x, y) dx.$$

Again, we can do that for more variables. The marginal pdf of X_j:

$$f_{X_j}(x_j) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} f_{X_1 X_2 \dots X_n}(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_{j-1} dx_{j+1} dx_n.$$

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Once again, this only makes sense if $f_X(x), f_Y(y) > 0$.

Say we are looking for $P(X \in A | Y \in B)$, we would calculate this as:

$$\frac{\int_A \int_B f_{XY}(x, y) dy dx}{\int_B f_Y(y) dy}.$$

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Example

A product is a mixture of two materials: let the volume of material 1 used be represented as X, and the volume of material 2 used be represented as Y. The joint probability density function of the two random variables is

$$f_{XY}(x,y) = \frac{2}{5}(2x+3y), \quad 0 \le x \le 1, 0 \le y \le 1.$$

What is the probability the first material has volume less than or equal to 0.5, and the second material has volume between 25% and 50%?

Answer: We are looking for $P(0 \le x \le 0.5, 0.25 \le y \le 0.5)$.

$$P(0 \le x \le 0.5, 0.25 \le y \le 0.5) = \int_0^{0.5}\int_{0.25}^{0.5} f_{XY}(x,y)dydx =$$

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$$= \frac{2}{5}\int_0^{0.5}\left(0.5x+\frac{9}{32}\right)dx = \frac{2}{5}\left(0.5\frac{x^2}{2}+\frac{9}{32}x\right)\Big|_0^{0.5} = \frac{13}{160}$$

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Example

A product is a mixture of two materials: let the volume of material 1 used be represented as X, and the volume of material 2 used be represented as Y. The joint probability density function of the two random variables is

$$f_{XY}(x,y) = \frac{2}{5}\left(2x+3y\right), \quad 0 \leq x \leq 1, 0 \leq y \leq 1.$$

What is the probability the first material has volume less than or equal to 0.5, and the second material has volume between 25% and 50%?

Answer: We are looking for $P(0 \leq x \leq 0.5, 0.25 \leq y \leq 0.5)$.

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With that in hand, we can calculate:

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