# Maximum likelihood estimation

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Lecture 18

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ISE | Industrial & Enterprise Systems Engineering
GRAINGER COLLEGE OF ENGINEERING

# Maximum likelihood estimation

Once more:

- Some population distributed with pdf $f(x)$.
- $f(x)$ depends on $m$ parameters, $\theta_1, \theta_2, \ldots, \theta_m$.
- Let $X_1, X_2, \ldots, X_n$ be a sample of that population.

Then:

**Definition**

The **likelihood function** of the sample is defined as

$$L(\theta) = f(X_1, \theta) \cdot f(X_2, \theta) \cdot \ldots \cdot f(X_n, \theta).$$

The **maximum likelihood estimator** $\hat{\Theta}$ is the value that maximizes the likelihood function.

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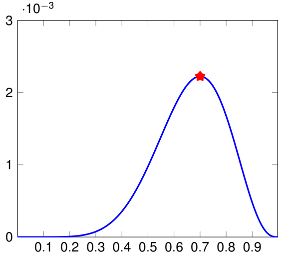
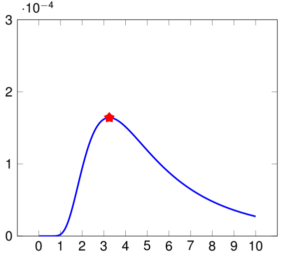
What is the full procedure?

1. Collect a sample $X_1, X_2, \dots, X_n$.
2. Build the likelihood function $L(\theta) = \prod_{i=1}^{n} f(X_i, \theta)$.
3. Find the maximum of $L$.
   - This can be done by visual inspection.
   - Or by taking the derivative(s) and equating them to 0 [1].
     - Specifically, if we are estimating $m$ parameters, we need to take $m$ partial derivatives, one for every parameter.
     - Then, we solve a system with $m$ equations and unknowns.




[1] If $L$ is concave.

# MLE example

**Example (Exponential distribution)**

*Times between accidents are exponentially distributed, but with unknown rate $\lambda$. We have observed the following sample of accident times: $X_1 = 5$ days, $X_2 = 10$ days, $X_3 = 8$ days, and $X_4 = 7$ days. What is the maximum likelihood estimator for $\lambda$?*

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Secondly, find the maximizer:

$$\frac{\partial L(\lambda)}{\partial \lambda} = 0 \implies 4\lambda^3 e^{-30\lambda} - 30 \cdot \lambda^4 e^{-30\lambda} \implies$$

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Hence, we estimate a rate of 1 accident every 7.5 days.

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# Log-likelihood function

What if we took the logarithm of the likelihood function

$$L(\theta) = f(X_1, \theta) \cdot f(X_2, \theta) \cdot \ldots \cdot f(X_n, \theta)?$$

**Definition**

The **log-likelihood function** of a sample $X_1, X_2, \ldots, X_n$ is defined as

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The **maximum likelihood estimator** $\hat{\Theta}$ can also be found as the maximum of the log-likelihood function.

It is useful as $\ln(a \cdot b) = \ln a + \ln(b)$ and it turns “difficult” multiplications to “easier” additions.

In general, recall that $\ln\left(\prod_i a_i\right) = \sum_i \ln(a_i)$.

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Again, find the maximizer:

$$\frac{\partial \ln L(\lambda)}{\partial \lambda} = 0 \implies (4 \ln \lambda - 30\lambda)' = 0 \implies$$
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**Example (Exponential distribution)**

*Times between accidents are exponentially distributed, but with unknown rate $\lambda$. We have observed the following sample of accident times: $X_1 = 5$ days, $X_2 = 10$ days, $X_3 = 8$ days, and $X_4 = 7$ days. What is the maximum likelihood estimator for $\lambda$?*

**Answer**: The log-likelihood function is:

$$\begin{aligned}\ln L(\lambda) &= \ln \lambda e^{-\lambda X_1} + \ln \lambda e^{-\lambda X_2} + \ln \lambda e^{-\lambda X_3} + \ln \lambda e^{-\lambda X_4} = \\ &= \ln \lambda - \lambda X_1 + \ln \lambda - \lambda X_2 + \ln \lambda - \lambda X_3 + \ln \lambda - \lambda X_4 = \\ &= 4 \ln \lambda - 30\lambda\end{aligned}$$

Again, find the maximizer:

$$\frac{\partial \ln L(\lambda)}{\partial \lambda} = 0 \implies (4 \ln \lambda - 30\lambda)' = 0 \implies$$

$$\frac{4}{\lambda} - 30 = 0 \implies \lambda = \frac{4}{30} = \frac{1}{7.5}.$$