# Counting

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Lecture 2

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ISE | Industrial & Enterprise Systems Engineering
GRAINGER COLLEGE OF ENGINEERING

# Last time..

During the previous lecture, we:

- defined random experiments, samples spaces, and events.
- introduced why data analysis under uncertainty is important.
- discussed set operations and how to calculate set cardinality.
  - union: $A \cup B$.
  - intersection: $A \cap B$.
  - complement: $\overline{A}$.
  - relative complement: $A \setminus B$.
  - set cardinality: $|A|$ ("size" of set $A$, # elements in the set).

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Today, we will discuss counting and how it relates to probabilities.

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# Probability

First, let us define what a probability is:

1. Frequentist view: probability is relative frequency.
2. Bayesian view: probability is “degree of belief”.

## Definition

With every event, we associate a number called *probability*: the likelihood that an event will happen. Probabilities follow three rules:

1. $P(E) \geq 0$.
2. If $E = S$, then $P(E) = 1$.
3. If $E_1, E_2, \ldots, E_m$ are $m$ mutually exclusive events then:

$$P\left(E_1 \cup E_2 \cup \ldots \cup E_m\right) = P(E_1) + P(E_2) + \ldots + P(E_m)$$

or:

$$P\left(\bigcup_{i=1}^{m} E_i\right) = \sum_{i=1}^{m} P(E_i).$$

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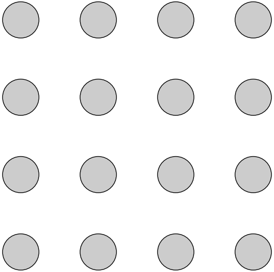
# Motivation: probabilities and counting

**Equally probable outcomes:**

- each outcome has probability $\frac{1}{|S|}$.
- **counting** all favorable outcomes and dividing by the total number of outcomes would reveal the likelihood.



Step 1: count all outcomes (16); Step 2: count all favorable outcomes (3); Step 3: calculate probability (3/16).

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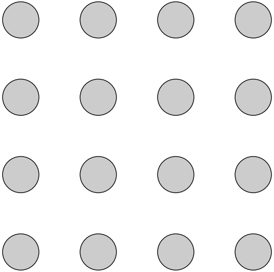

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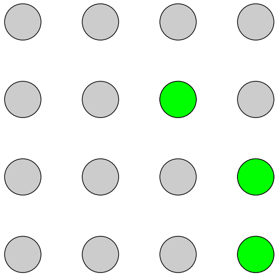

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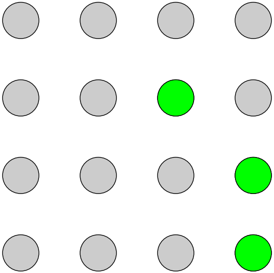

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# The multiplication rule

When our set of outcomes comes from a sequence of $k$ steps, each of them with $n_i$, $i = 1, \ldots k$ options (i.e., $n_1$ options in step 1, $n_2$ options in step 2, and so on), then the number of outcomes is:

$$\boxed{n_1 \cdot n_2 \cdot \ldots \cdot n_k.}$$

Two key observations:

1. at each step $i$, we can have exactly one of the $n_i$ options.
2. the order does not matter.

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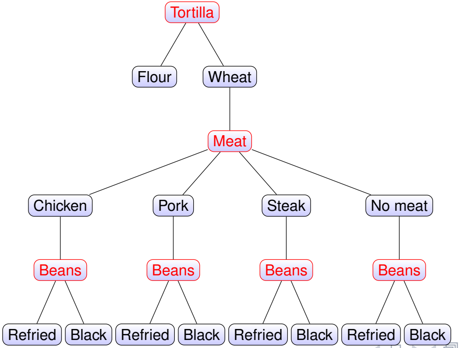

# Permutations

**Definition**

A permutation is an **ordered** sequence of elements selected from some set.

We define two types of permutations for a set with $n$ elements:

1. using all $n$ elements in a set.
   The number of permutations we can have using $n$ outcomes is
   $$P_n = n!\,[1]$$
2. using only $r < n$ elements from the set.
   The number of permutations using $r < n$ outcomes is
   $$P_{n,r} = \frac{n!}{(n-r)!}$$

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# Distinguishable permutations

When we have multiple elements that are indistinguishable from one another, we calculate permutations of $n$ elements a little differently.

- If we have $k$ types of elements with $n_i$ objects of type $i$ $(i = 1, \ldots, k)$ such that $\sum_{i=1}^{k} n_i = n$, then the number of distinguishable permutations is:

$$\boxed{\frac{n!}{n_1! \cdot n_2! \cdot \ldots n_k!}}$$

**Example**

*How many 4-letter words (even nonsensical) can we construct using $2 \times A$, $1 \times B$, and $1 \times C$?*

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Twelve: *AABC*, *AACB*, *ABAC*, *ABCA*, *ACAB*, *ACBA*, *BAAC*, *BACA*, *BCAA*, *CAAB*, *CABA*, *CBAA*

# Combinations

We finish today's discussion on counting rules with **combinations**.

In all of the examples we have seen so far (PIN, leadership team, Scrabble) *order matters*. Often, though, we do not care about it.

- Creating a group of 4 people for a class project.
- Checking the numbers on two dice.
- Picking the winning numbers in a lottery.

**Definition**

A **combination** is an unordered subset of $r < n$ elements selected from a set with $n$ elements.

The number of all possible combinations is calculated by:

$$C_{n,r} = \binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}$$ [2]

[2] $\binom{n}{r}$ is also read as "$n$ choose $r$".

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# From counting to probabilities

**Example**

*A PIN consists of 4 digits (from 0-9). What is the probability that the correct PIN to unlock the phone is 1234?*

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*A PIN consists of 4 digits (from 0-9), but using a keylogger you have been able to see that the pressed buttons were "1", "2", "3", and "4". You have no idea about what the correct order is. What is the probability that the correct PIN to unlock the phone is 1234?*

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*A PIN consists of 4 digits (from 0-9), but you are not allowed to use the same digit more than once. What is the probability the correct PIN to unlock the phone is 1234?*

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permutation of 4 numbers out of 10 elements: $\frac{1}{\frac{10!}{(10-4)!}} = \frac{1}{5040}$

# From counting to probabilities

**Example**

*In a deck of cards, there are 52 cards, 12 of which correspond to "face" cards (J, Q, K of 4 suits). You pick 3 cards at random. What is the probability that all 3 are "face" cards? What is the probability that all 3 are not "face" cards?*

1. How many ways are there to select 3 cards out of a total of 52 cards?

2. How many ways are there to have 3 face cards out of the 12 available ones?

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1. How many ways are there to select 3 cards out of a total of 52 cards?

   combination of 3 elements out of 52: $C_{52,3} = \binom{52}{3} = 22100.$

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   combination of 3 elements out of 12: $C_{12,3} = \binom{12}{3} = 220.$

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   combination of 3 elements out of 12: $C_{12,3} = \binom{12}{3} = 220$.

Combining, the probability is $\frac{220}{22100} \approx 0.01$.