Hypothesis testing for means and variances

Chrysafis Vogiatzis

Department of Industrial and Enterprise Systems Engineering
University of Illinois at Urbana-Champaign

Lecture 26-27

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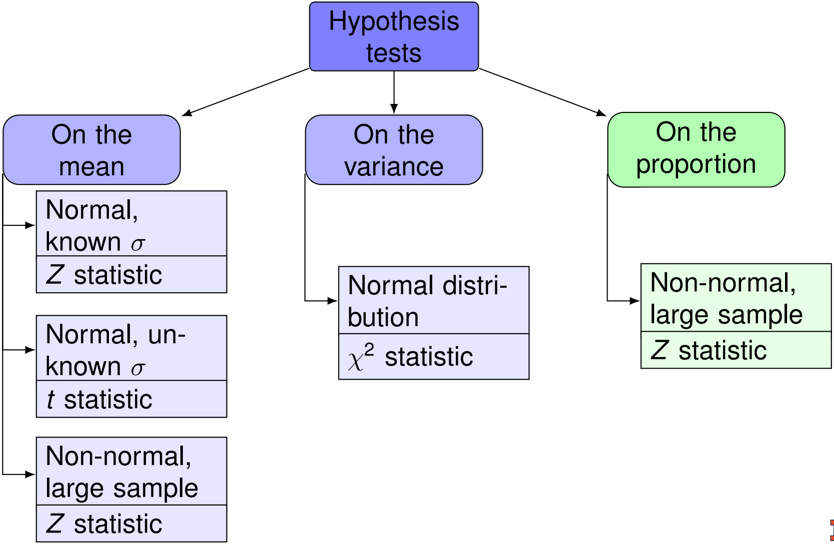
Overview



Hypothesis tests
On the mean
On the variance
On the proportion
Normal, known σ
Z statistic
Normal, unknown σ
t statistic
Non-normal, large sample
Z statistic
Normal distribution
χ^2 statistic
Non-normal, large sample
Z statistic

Proportions: the procedure

Null hypothesis:	Test statistic:	Distribution:
$H_0 : p = p_0.$	$Z_0 = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}.$	$Z_0 \sim \mathcal{N}(0, 1).$

H_1	Rejection region	P-value
$p \neq p_0$	$\lvert Z_0 \rvert > z_{\alpha/2}$	$2 \cdot (1 - \Phi(\lvert Z_0 \rvert))$
$p > p_0$	$Z_0 > z_\alpha$	$1 - \Phi(Z_0)$
$p < p_0$	$Z_0 < -z_\alpha$	$\Phi(Z_0)$

Reject if Z_0 or $\hat{p}$ falls in the rejection region or if P-value $< \alpha$.

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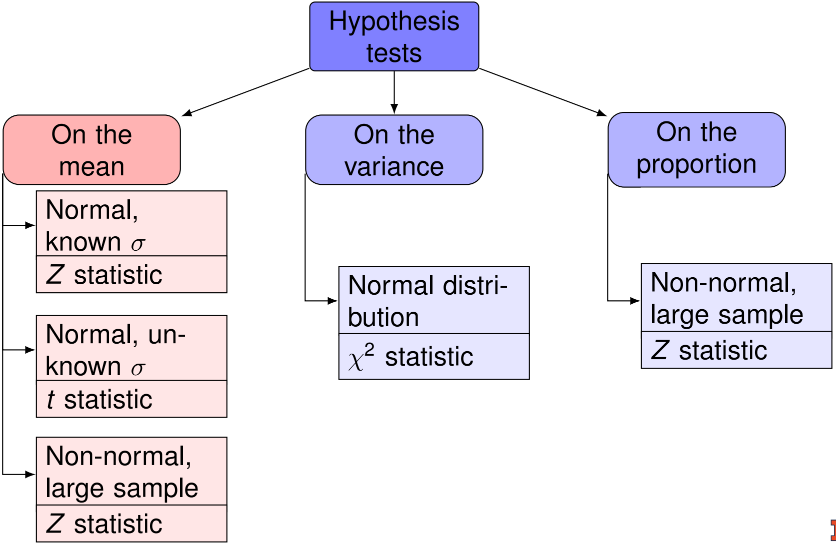
Hypothesis testing for means



Hypothesis testing for means of normally distributed populations with known variance

Null hypothesis:

$$H_0 : \mu = \mu_0.$$

Test statistic:

$$Z_0 = \frac{\overline{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}.$$

Distribution:

$$Z_0 \sim \mathcal{N}(0, 1).$$

H_1	Rejection region	P-value
$\mu \neq \mu_0$	$\lvert Z_0 \rvert > z_{\alpha/2}$	$2 \cdot (1 - \Phi(\lvert Z_0 \rvert))$
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Reject if Z_0 or $\overline{X}$ falls in the rejection region or if P-value $< \alpha$.

Hypothesis testing for means of normally distributed populations with unknown variance

Null hypothesis:	Test statistic:	Distribution:
$H_0 : \mu = \mu_0.$	$T_0 = \frac{\overline{X} - \mu_0}{\frac{s}{\sqrt{n}}}.$	$T_0 \sim T_{n-1}.$

H_1	Rejection region	P-value
$\mu \neq \mu_0$	$\lvert T_0 \rvert > t_{\alpha/2,n-1}$	$2 \cdot (1 - T_{n-1}(\lvert T_0 \rvert))$
$\mu > \mu_0$	$T_0 > t_{\alpha,n-1}$	$1 - T_{n-1}(T_0)$
$\mu < \mu_0$	$T_0 < -t_{\alpha,n-1}$	$T_{n-1}(T_0)$

Reject if T_0 or $\overline{X}$ falls in the rejection region or if P-value $< \alpha$.

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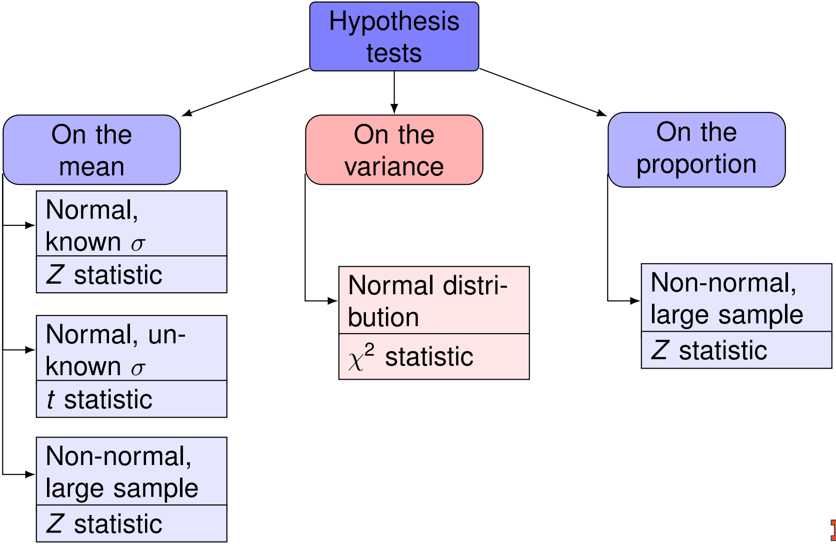
Hypothesis testing for means of not normally distributed populations

Null hypothesis:	Test statistic:	Distribution:
$H_0 : \mu = \mu_0.$	$Z_0 = \dfrac{\overline{X} - \mu_0}{\frac{s}{\sqrt{n}}}.$	$Z_0 \sim \mathcal{N}(0, 1).$

H_1	Rejection region	P-value
$\mu \neq \mu_0$	$\lvert Z_0 \rvert > z_{\alpha/2}$	$2 \cdot (1 - \Phi(\lvert Z_0 \rvert))$
$\mu > \mu_0$	$Z_0 > z_\alpha$	$1 - \Phi(Z_0)$
$\mu < \mu_0$	$Z_0 < -z_\alpha$	$\Phi(Z_0)$

Reject if Z_0 or $\overline{X}$ falls in the rejection region or if P-value $< \alpha$.

Hypothesis testing for variances



Hypothesis testing for variances of normally distributed populations

Null hypothesis:	Test statistic:	Distribution:
$H_0 : \sigma^2 = \sigma_0^2.$	$\chi_0^2 = \dfrac{(n-1)\,s^2}{\sigma_0^2}.$	$\chi_0^2 \sim \chi_{n-1}^2.$

H_1	Rejection region	CI region
$\sigma^2 \neq \sigma_0$	$\chi_0^2 > \chi^2_{\alpha/2,n-1}$ $\chi_0^2 < \chi^2_{1-\alpha/2,n-1}$	$\left[\dfrac{(n-1)\sigma_0^2}{\chi^2_{\alpha/2,n-1}}, \dfrac{(n-1)\sigma_0^2}{\chi^2_{1-\alpha/2,n-1}}\right]$
$\sigma^2 > \sigma_0$	$\chi_0^2 > \chi^2_{\alpha,n-1}$	$\left[\dfrac{(n-1)\sigma_0^2}{\chi^2_{\alpha,n-1}}, +\infty\right)$
$\sigma^2 < \sigma_0$	$\chi_0^2 < \chi^2_{1-\alpha,n-1}$	$\left(-\infty, \dfrac{(n-1)\sigma_0^2}{\chi^2_{1-\alpha,n-1}}\right]$

Reject if χ_0^2 or σ_0^2 falls in the rejection region.

Example

A call center is concerned that call durations for a customer service representative are too ***erratic****: high variations is call durations can lead to customer dissatisfaction who have to wait longer for a resolution. The company has collected data from* $n = 24$ *randomly selected phone calls from that specific customer representative and calculated that* $s = 5$ *minutes.*

1. *Is there enough evidence to suggest that* $\sigma = 4$ *minutes? Use* $\alpha = 0.05$.
2. *Assume that we do not care about the standard deviation being lower than 4 minutes; instead, we are only interested if the standard deviation is higher than that. Is there enough evidence to suggest that* $\sigma = 4$ *minutes or is it higher than that? Again, you may use that* $\alpha = 0.05$.

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Solution to the example

First, set up your hypothesis:

$$H_0 : \sigma^2 = 16$$
$$H_1 : \sigma^2 \neq 16.$$

- Calculate $\chi_0^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{23 \cdot 5^2}{16} = 35.94.$
- Find the critical values for $\chi^2_{0.025,23}$ and $\chi^2_{0.975,23}$ as 38.076 and 11.689, respectively.
- **Fail to reject** as $\chi^2_{0.975,23} \leq \chi^2_0 \leq \chi^2_{0.025,23}$.

For the second part, set up the hypothesis as:

$$H_0 : \sigma^2 = 16$$
$$H_1 : \sigma^2 > 16.$$

- The test statistic is still $\chi_0^2 = 35.94$.
- However now we are only looking for $\chi^2_{\alpha,n-1} = \chi^2_{0.05,23} = 35.172$.
- **Reject** then as $\chi_0^2 > \chi^2_{0.05,23}$.

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