Hypothesis testing for two populations

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Lectures 28-29

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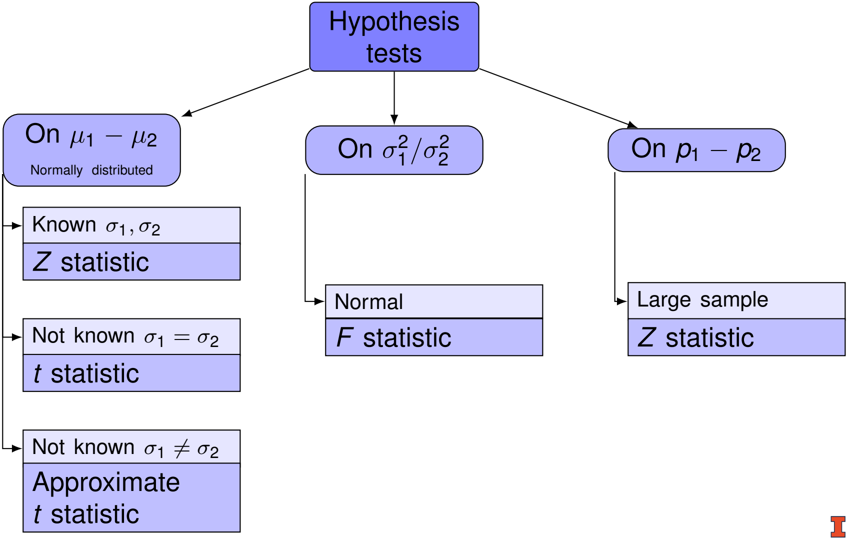
Overview



Hypothesis tests
On $\mu_1 - \mu_2$
Normally distributed
Known σ_1, σ_2
Z statistic
Not known $\sigma_1 = \sigma_2$
t statistic
Not known $\sigma_1 \neq \sigma_2$
Approximate t statistic
On σ_1^2/σ_2^2
Normal
F statistic
On $p_1 - p_2$
Large sample
Z statistic

Hypothesis testing for $\mu_1 - \mu_2$

What if we are interested in testing what the difference between two means is?

- μ_1: unknown mean of population 1.
- μ_2: unknown mean of population 1.

We distinguish between:

1. normally distributed populations with known variances σ_1^2, σ_2^2.
2. normally distributed populations with unknown variances that are known to be equal, that is unknown $\sigma_1^2 = \sigma_2^2$.
3. normally distributed populations with unknown variances that are not known to be equal, that is unknown $\sigma_1^2 \neq \sigma_2^2$.

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Means with known σ_1, σ_2

Null hypothesis: | Test statistic: | Distribution:

$$H_0 : \mu_1 - \mu_2 = \Delta_0.$$

$$Z_0 = \frac{(\overline{X}_1 - \overline{X}_2) - \Delta_0}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}.$$

$$Z_0 \sim \mathcal{N}(0, 1).$$

H_1	Rejection region	P-value
$\mu_1 - \mu_2 \neq \Delta_0$	$\lvert Z_0 \rvert > z_{\alpha/2}$	$2 \cdot (1 - \Phi(\lvert Z_0 \rvert))$
$\mu_1 - \mu_2 > \Delta_0$	$Z_0 > z_\alpha$	$1 - \Phi(Z_0)$
$\mu_1 - \mu_2 < \Delta_0$	$Z_0 < -z_\alpha$	$\Phi(Z_0)$

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Means with unknown $\sigma_1 = \sigma_2 = \sigma$

First, define a pooled estimator for the variance:

$$s_p^2 = \frac{(n_1 - 1)\, s_1^2 + (n_2 - 1)\, s_2^2}{n_1 + n_2 - 2}.$$

Null hypothesis: $H_0 : \mu_1 - \mu_2 = \Delta_0.$

Test statistic: $$T_0 = \frac{(\overline{X}_1 - \overline{X}_2) - \Delta_0}{s_p\sqrt{1/n_1 + 1/n_2}}.$$

Distribution: $T_0 \sim T_{n_1+n_2-2}.$

H_1	Rejection region	P-value
$\mu_1 - \mu_2 \neq \Delta_0$	$\lvert T_0 \rvert > t_{\alpha/2, n_1+n_2-2}$	$2 \cdot (1 - T_{n_1+n_2-2}(\lvert T_0 \rvert))$
$\mu_1 - \mu_2 > \Delta_0$	$T_0 > t_{\alpha, n_1+n_2-2}$	$1 - T_{n_1+n_2-2}(T_0)$
$\mu_1 - \mu_2 < \Delta_0$	$T_0 < -t_{\alpha, n_1+n_2-2}$	$T_{n_1+n_2-2}(T_0)$

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Means with unknown $\sigma_1 \neq \sigma_2$

We can no longer calculate a pooled estimator for the variance: instead, we need to use the two sample standard deviations in the place of the actual ones.

We can no longer calculate the actual degrees of freedom: we estimate the approximate degrees of freedom as

$$v = \frac{\left(s_1^2/n_1 + s_2^2/n_2\right)^2}{\frac{\left(s_1^2/n_1\right)^2}{n_1-1} + \frac{\left(s_2^2/n_2\right)^2}{n_2-1}}.$$

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$$T_0 = \frac{(\overline{X}_1 - \overline{X}_2) - \Delta_0}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}.$$

$$T_0 \sim T_v.$$

H_1	Rejection region	P-value
$\mu_1 - \mu_2 \neq \Delta_0$	$\lvert T_0 \rvert > t_{\alpha/2,v}$	$2 \cdot (1 - T_v(\lvert T_0 \rvert))$
$\mu_1 - \mu_2 > \Delta_0$	$T_0 > t_{\alpha,v}$	$1 - T_v(T_0)$
$\mu_1 - \mu_2 < \Delta_0$	$T_0 < -t_{\alpha,v}$	$T_v(T_0)$

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Ratio of variances

For two normally distributed populations with unknown variances σ_1^2 and σ_2^2, we have:

Null hypothesis: $H_0 : \sigma_1^2 = \sigma_2^2$.

Test statistic: $F_0 = \frac{s_1^2}{s_2^2}$.

Distribution: $F_0 \sim F_{n_1-1,n_2-1}$.

H_1	Rejection region
$\sigma_1^2 \neq \sigma_2^2$	$F_0 > f_{\alpha/2,n_1-1,n_2-1}$ or
	$F_0 < f_{1-\alpha/2,n_1-1,n_2-1}$
$\sigma_1^2 > \sigma_2^2$	$F_0 > f_{\alpha,n_1-1,n_2-1}$
$\sigma_1^2 < \sigma_2^2$	$F_0 < f_{1-\alpha,n_1-1,n_2-1}$

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Difference in proportions

How about for the difference in the proportions of two populations $p_1 - p_2$?

- Let $n_i, \hat{p}_i$ be the sample sizes and observed proportions.
- Assume that $n_i p_i$ and $n_i(1 - p_i)$ are ≥ 30.

Define a pooled proportion estimator:

$$\hat{p} = \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2}.$$

Why do we need this? Well, we somehow need to quantify the variance of the combined sample...

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