Basic probability theory

Chrysafis Vogiatzis

Department of Industrial and Enterprise Systems Engineering
University of Illinois at Urbana-Champaign

Lecture 3

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GRAINGER COLLEGE OF ENGINEERING

Last time..

- We discussed different ways to count.
  - **Multiplication rule:** when tasked with making k choices, each of them with n_i options ($i = 1, \ldots, k$): $n_1 \cdot n_2 \cdot \ldots \cdot n_k$
  - **Permutation of all elements of a set:** when having n elements to arrange in an ordered fashion: $P_n = n \cdot (n-1) \cdot \ldots \cdot 1 = n!$
  - **Permutation of part of the elements of a set:** when picking $r < n$ elements to arrange in an ordered fashion:
    $$P_{n,r} = n \cdot (n-1) \cdot \ldots \cdot (n-r) = \frac{n!}{(n-r)!}$$
  - **Permutation of groups of indistinguishable elements:** when faced with k groups of elements, each with n_i items, then the distinguishable permutations are: $\frac{n!}{n_1! \cdot n_2 \cdot \ldots \cdot n_k!}$
  - **Combinations:** when picking $r < n$ elements to arrange in an unordered fashion: $C_{n,r} = \binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}$
- We tied counting to quantifying probabilities in the case of equally probable events.

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Probability

As a reminder, probability is a real number that quantifies the likelihood of an event happening.

1. $P(E) \geq 0$.
2. If $E = S$, then $P(E) = 1$.
3. If $E_1, E_2, \ldots, E_m$ are m mutually exclusive events then:

$$P\left(E_1 \cup E_2 \cup \ldots \cup E_m\right) = P(E_1) + P(E_2) + \ldots + P(E_m)$$

or:

$$P\left(\bigcup_{i=1}^{m} E_i\right) = \sum_{i=1}^{m} P(E_i).$$

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The three axioms above immediately give us some very useful properties:

- Probability is always between 0 and 1.
- $P(\overline{E}) = 1 - P(E)$.
- If $E_1 \subseteq E_2$, then $P(E_1) \leq P(E_2)$.

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Union and intersection of events

Recall that for any two events E_1, E_2, we have:

- $E_1 \cup E_2$: at least one of E_1, E_2 should happen.
- $E_1 \cap E_2$: both E_1 and E_2 should happen.

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From this definition, we may deduce that:

$$P(E_1) \leq P(E_1 \cup E_2) \qquad P(E_2) \leq P(E_1 \cup E_2)$$
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Additionally:

$$P(E_1 \cup E_2) \leq P(E_1) + P(E_2)$$

Calculating $P(E_1 \cup E_2)$: a small example

Example

Recall the grades from 3 different professors for the same class.

Letter Grade	Professor 1	Professor 2	Professor 3	Total
A	108	20	30	158
B	44	49	46	139
C	11	15	15	41
D	0	1	8	9
Total	163	85	99	347

You call on 1 student out of the 347, what is the probability:

1. E_1: *you pick a student from Professor 1's class?*

   $P(E_1) = 163/347 = 0.4697.$

2. E_2: *you pick a student who received an A in the class?*

   $P(E_2) = 158/347 = 0.4553.$

3. $E_1 \cap E_2$: *you pick a student who was both in Professor 1's class and received an A in the class?*

   $P(E_1 \cap E_2) = 108/347 = 0.3112.$

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Overall:

$$\boxed{P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)}$$

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See the worksheet for the union of more than 2 events.

Conditional probabilities

Sometimes, we are interested in finding the likelihood of an event *under certain circumstances*. We formally define this as **the probability that an event E_2 happens given that event E_1 has happened** and we write[1]:

$$P(E_2|E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)}.$$

If two events are *mutually exclusive*, we have that $P(E_2|E_1) = 0$.

Based on the formula for calculating conditional probabilities, we also have the multiplication rule for probabilities:

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2|E_1).$$

[1] Be careful: we need $P(E_1) > 0$ for this to make sense.

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Independent events

Two events are **independent** if knowledge that one has happened does not affect the probability of the other, that is:

$$P(E_2|E_1) = P(E_2) \text{ or } P(E_1|E_2) = P(E_1).$$

Equivalently, two events are **independent** if

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2).$$

This is easily shown by using the fact that

$$\overbrace{P(E_2|E_1)}^{P(E_2)} = \frac{P(E_1 \cap E_2)}{P(E_1)} \implies P(E_1 \cap E_2) = P(E_1) \cdot P(E_2).$$

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Example: Let's play some cards

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Consider a deck of 52 cards, with 13 cards from each suit: **spades ♠**, **hearts ♡**, **diamonds ♢**, **clubs ♣**.

1. What is the probability that you draw a 2? There are 4 twos in the deck: $4/52 = 1/13$.
2. What is the probability that you draw a diamond? There are 13 of each suit in the deck: $13/52 = 1/4$.
3. What is the probability that you draw a diamond or a spade? The suits are mutually exclusive: $1/4 + 1/4 = 1/2$.
4. What is the probability that you draw a diamond and a spade? The suits are mutually exclusive: 0.
5. Are "drawing a diamond" and "drawing a red card" independent events? They are not. Knowing we drew a red card, changes the probability of picking a diamond from $1/4$ to $1/2$.
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