Discrete random variables

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Lectures 5 and 6

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Last time..

- Law of total probability.

$$P(B) = P(A) \cdot P(B|A) + P(\bar{A}) \cdot P(B|\bar{A}).$$

- Bayes' theorem.
  - states S_i, with known $P(S_i)$;
  - outcomes O_j, with known $P(O_j|S_i)$.

$$P(S_i|O_j) = \frac{P(S_i) \cdot P(O_j|S_i)}{\sum_{k=1}^{n} P(S_k) \cdot P(O_j|S_k)}.$$

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Today, we will introduce random variables and, specifically, discrete random variable probability distributions.

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Random variables

Definition

A random variable is a real-valued function defined over the sample space.

Definition

A random variable is a function that associates a number with each element of the sample space.

We separate the discussion between discrete and continuous random variables.

- Discrete random variables takes a countable (finite or infinite) number of discrete values (e.g., side of a die, number of customers).
- Continuous random variables can take any real-value (e.g., time until next bus arrives, lifetime of a light bulb).

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Probability distributions and functions

Definition

A probability distribution is a mathematical function that ties probabilities to the values that a random variable is allowed to take.

We distinguish between two types of distribution functions:

1. probability mass/distribution functions (pmf/pdf).
2. cumulative distribution functions (cdf).

These two functions can describe a probability distribution.

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Discrete random variables

pmf: $p(x) = P(X = x)$: the probability that random variable X is equal to some value x.

1. $p(x_i) = P(X = x_i)$, for every outcome x_i, $i = 1, \ldots, n$.
2. $p(x_i) \geq 0$.
3. $\sum_{i=1}^{n} p(x_i) = 1$

cdf: $F(x) = P(X \leq x) = \sum_{y: y \leq x} P(X = y)$: the probability that random variable X is up to some value x.

1. $0 \leq F(x) \leq 1$.
2. If $x \leq y$, then $F(x) \leq F(y)$.

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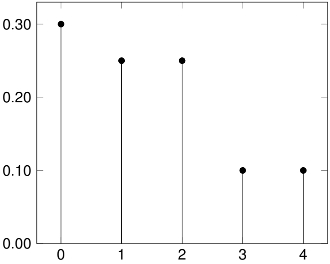

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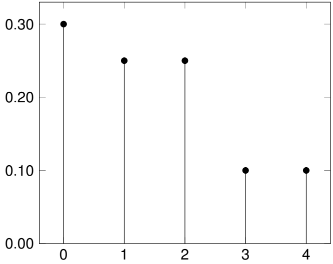

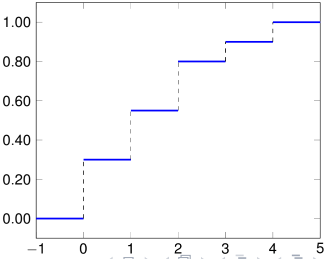

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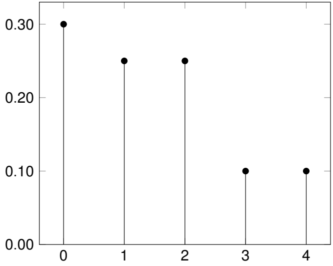
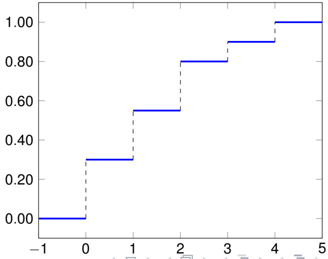
$P(a \leq X \leq b) = F(b) - F(a)$.





Bernoulli distributed random variables

- Consider a **single** experiment/trial with only two outcomes: **success** with probability p or **failure** with probability $q = 1 - p$.
- Now define a random variable X:

$$X = \begin{cases} 0, & \text{if the experiment failed;} \\ 1, & \text{if the experiment succeeded.} \end{cases}$$

  - Will the next coin toss be a heads (success) or a tail (failure)?
  - Will it rain (success) or not (failure)?
  - Will the next patient be cured (success) or not (failure)?

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$$\text{pmf:} \quad p(0) = P(X = 0) = q = 1 - p$$
$$p(1) = P(X = 1) = p$$

$$\text{cdf:} \quad F(x) = \begin{cases} 0, & x < 0 \\ 1 - p, & x < 1 \\ 1, & x \geq 1 \end{cases}$$

The binomial distribution

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What if.. we could run and observe multiple experiments?

- Multiple coin tosses? A control group of 100 patients? A five game series?

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Formally:

- n independent trials;
- each one is a success with probability p and a failure with probability $q = 1 - p$;

each one is a Bernoulli random variable!

- the number of successes in n tries is a **binomially distributed random variable** $X = binom(n, p)$.

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The binomial distribution from an example

Example

What is the probability that there will be exactly 2 heads in $n = 3$ tosses of a "fair" ($p = 0.5$ heads, $1 - p = 0.5$ tails) coin?

- How many different scenarios are there?
  - $\{H, H, H\}, \{H, H, T\}, \{H, T, H\}, \{H, T, T\}, \{T, H, H\}, \{T, H, T\}, \{T, T, H\}, \{T, T, T\}$.
- $Pr\{X = 2\} = \frac{3}{8}$.
- What if we had 10, 100, 1000 coin tosses?

For $x = 0, 1, \ldots, n$:

$$p(x) = P(X = x) = \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x}.$$

$$F(x) = P(X \le x) = \sum_{y: y \le x} P(X = y) = \sum_{y: y \le x} \binom{n}{y} \cdot p^y \cdot (1-p)^{n-y}.$$

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Still based on Bernoulli random variables:

- How many attempts until a success?
- For example, what is the probability that the first *heads* is seen in the third coin toss?
- The probability that the first success is seen after x trials is:

$$p(x) = P(X = x) = (1 - p)^{x-1} \cdot p.$$

why?

Example

What is the probability that the first heads is seen at the third *coin toss of a "fair" coin ($p = 0.5$)? How about at the* tenth *coin toss?*

- $P(X = 3) = (1 - p)^2 \cdot p = 0.125.$
- $Pr(X = 10) = (1 - p)^9 \cdot p = 0.0009765625 \approx 0.001.$

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The hypergeometric distribution

What if..

- we had N elements;
- $K \leq N$ of them are successes (the remaining $N - K$ are failures);
- we picked a sample of size n from them;
- what is the probability we get x successes in that sample of n?

We have actually seen this problem before...

$$\text{pmf: } p(x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

Example

An urn contains 40 black and 10 red balls. You pick at random a sample of five balls from the urn. Let X be the number of black balls in the sample. What is the probability that $X = 3$?

Answer: We have $N = 50, K = 40, k = 3, n = 10$: $\frac{\binom{40}{3}\binom{10}{2}}{\binom{50}{5}} = 0.2098$.

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The Poisson distribution

Definition

A random variable X taking values $0, 1, 2, \ldots$ is a Poisson random variable with parameter (**rate**) $\lambda > 0$ if:

$$p(x) = P(X = x) = e^{-\lambda} \frac{\lambda^x}{x!}.$$

Poisson random variables have a wide, *wide* array of applications. They have be used to model:

- the number of website requests per second.
- the number of shark attacks in California every year.
- the number of home runs in a baseball series.
- the number of patients arriving in an emergency department every night.

Two assumptions:

1. independence.
2. homogeneity.

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The uniform distribution

Think of a discrete random variable X that can take any of n different outcomes $x_i, i = 1, \ldots, n$.

- If all n outcomes are equally probable, then we have a uniform random variable.
- Each of the outcomes has a probability of $p_i = P(X = x_i) = \frac{1}{n}$.
- In a special case, the discrete random variable takes values in $[a, b]$. Then, the pmf is $p_i = \frac{1}{b-a+1}$, for all $i \in [a, b]$.

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Discrete random variables: summary

Table: A summary of all results from Lectures 5 and 6.

Name	Parameters	Values	pmf
Bernoulli	$0 < p < 1$	$\{0, 1\}$	$p(0) = 1 - p$ $p(1) = p$
Binomial	$0 < p < 1,\ n \geq 0$	$\{0, 1, \ldots, n\}$	$p(x) = \binom{n}{x} p^x \cdot (1-p)^{n-x}$
Geometric	$0 < p < 1$	$\{1, 2, \ldots\}$	$p(x) = (1-p)^{x-1} \cdot p$
Hypergeometric	$N, K, n \geq 0$	$\{1, 2, \ldots\}$	$p(x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$
Poisson	$\lambda > 0$	$\{0, 1, \ldots\}$	$p(x) = e^{-\lambda} \frac{\lambda^x}{x!}$
Uniform	-	$[a, b]$	$p(x) = \frac{1}{b - a + 1}$

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Discrete random variables: examples

Example

In a game of tennis, Player A wins 70% of the points that are played; Player B wins the remaining 30% of the points. What is the probability Player B wins the next point?

Bernoulli: 0.3.

Example

In the same game of tennis, what is the probability that Player B wins 2 of the next 5 points?

binomial: $\binom{5}{2} \cdot 0.3^2 \cdot 0.7^3 = 0.3087$.

Example

Continuing with tennis, what is the probability that the first point Player B wins is the 5th one between the two players?

geometric: $0.7^4 \cdot 0.3 = 0.07203$.

Discrete random variables: more examples

Example

In the game of poker, five cards of the same suit make a flush. Assuming there are 52 cards with 4 suits of 13 cards each, what is the probability of being handed a flush of ♡?

Hypergeometric with $N = 52$, $K = 13$, $n = 5$, $x = 5$: $\dfrac{\binom{13}{5}\binom{39}{0}}{\binom{52}{5}} = 0.000495$.

Example

CA has an average of 1.8 shark attacks per year. If shark attacks are independent and the attack rate is homogeneous, what is the probability of a sharkless year in CA? How about seeing up to 2 attacks in the next 5 years?

Poisson with rate $\lambda = 1.8$ and $\mu = 9$: $e^{-1.8} \cdot \frac{1.8^0}{0!} = 0.1653$ and $e^{-9} \cdot \frac{9^0}{0!} + e^{-9} \cdot \frac{9^1}{1!} + e^{-9} \cdot \frac{9^2}{2!} = 0.0062$.

Example

Families arriving to a theater have 2, 3, 4, 5, or 6 members. What is the pmf of the number of members in a family coming to the theater today?

uniform: $p(x_i) = \frac{1}{6-2+1} = \frac{1}{5}$.