# Continuous random variables

Chrysafis Vogiatzis

Department of Industrial and Enterprise Systems Engineering
University of Illinois at Urbana-Champaign

Lecture 7a

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ISE | Industrial & Enterprise Systems Engineering
GRAINGER COLLEGE OF ENGINEERING

# Last time..

- Random variables.
- Discrete random variables.
  - Bernoulli, binomial, geometric;
  - Hypergeometric;
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This lecture is divided into two smaller videos. This is the first one focusing on continuous random variables fundamentals.

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# Continuous random variables

### Definition

A random variable is **continuous** if it can take uncountably many values such that there exists some function $f(x)$ called a **probability density function** defined over real values $(-\infty, +\infty)$ such that:

- $f(x) \geq 0$.
- $\int_{-\infty}^{+\infty} f(x)dx = 1$.
- $P(X \in B) = \int_B f(x)dx$

### Example

*What is the probability that random variable $X$ with pdf $f(x)$ is between 0 and 10?*

**Answer:** $P(0 \leq X \leq 10) = \int_0^{10} f(x)dx$.

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# Continuous random variable functions

**pdf:** $f(x)$: *relative likelihood* that random variable $X$ is equal to some value $x$.

1. $f(x) = 0$ implies that value $x$ cannot happen.
2. $f(x) \geq 0$.
3. $\int_{-\infty}^{+\infty} f(x)dx = 1$.

**cdf:** $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)dy$: the probability that random variable $X$ is up to some value $x$.

1. $0 \leq F(x) \leq 1$.
2. If $x \leq y$, then $F(x) \leq F(y)$ and also $P(a \leq X \leq b) = F(b) - F(a)$.
3. By definition $f(x) = F'(x)$.

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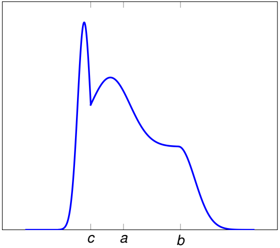





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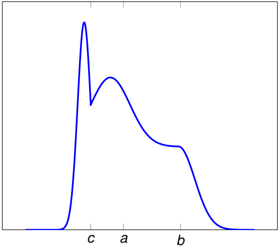


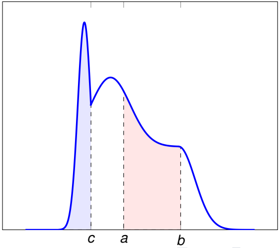

# The probability density function: a small example

**Example**

*Assume that $X$ is a continuous random variable with pdf $f(x) = c \cdot (1 + 0.1 \cdot x)$, for values of $x$ such that $-1 \leq x \leq 1$. For which value of $c$ is this a valid pdf?*

**Answer**: First, we observe whether $f(x) \geq 0$ for all values that $x$ can take. The smallest value that $x$ can take is $-1$, at which point we have $f(-1) = 0.9 \cdot c$. This is non-negative if $c \geq 0$.

Secondly, we know that $\int_{-\infty}^{+\infty} f(x)dx = 1$.

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$$\int_{-\infty}^{+\infty} f(x)dx = 1 \implies \int_{-\infty}^{+\infty} c \cdot (1 + 0.1 \cdot x)dx = 1 \implies$$

$$\implies cx\Big|_{-1}^{1} + c\frac{0.1 \cdot x^2}{2}\Bigg|_{-1}^{1} = 1 \implies 2c = 1 \implies c = \frac{1}{2}$$

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## Example

*For the previous random variable X, with pdf $f(x) = 0.5 \cdot (1 + 0.1 \cdot x), -1 \le x \le 1$, what is the cumulative distribution function? With that in hand, what is the probability that:*

a) $X \le 0$?  b) $X = 0$?  c) $-0.1 \le X \le 0.1$?

**Answer**: By definition, $F(x) = \int_{-1}^{x} f(y)dy$:

$$F(x) = \int_{-1}^{x} f(y)dy = \int_{-1}^{x} 0.5 \cdot (1 + 0.1 \cdot y)dy = 0.5 \cdot x + 0.05 \cdot \frac{x^2}{2} + 0.475.$$

For the probability questions now:

a) Two ways to calculate this:
- $P(X \le 0) = F(0) = 0.475$.
- $P(X \le 0) = \int_{-1}^{0} f(x)dx = \int_{-1}^{0} 0.5 \cdot (1 + 0.1 \cdot x)dx = 0.475$.

b) Trick question: it is zero! $\int_{0}^{0} f(x)dx = 0$, or $F(0) - F(0) = 0$.

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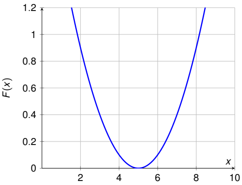
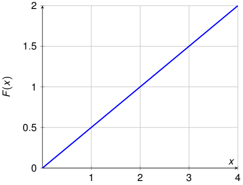
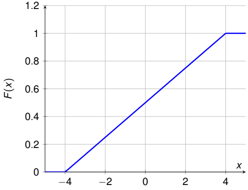



No. $F(x)$ has to be non-decreasing.




No. $F(x)$ has to be $\leq 1$.

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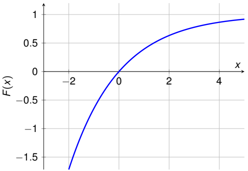
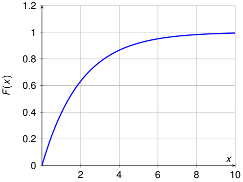



No. $F(x)$ has to be $\geq 0$.

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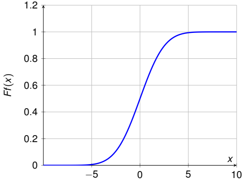


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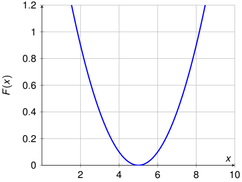


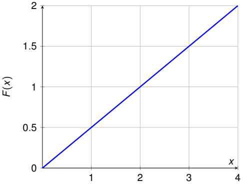


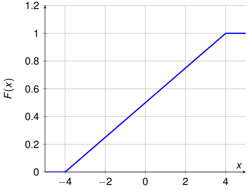


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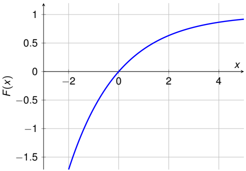


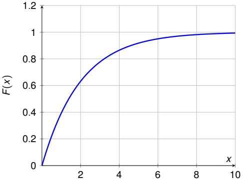


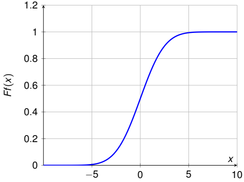


No. $F(x)$ has to be $\geq 0$.

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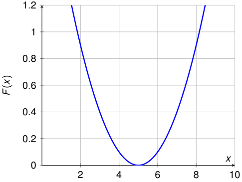


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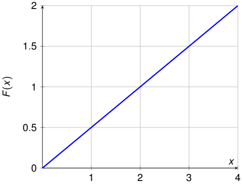


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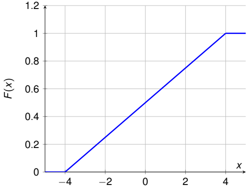


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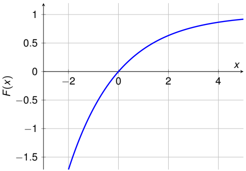


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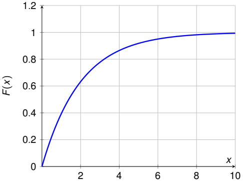


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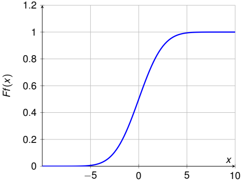


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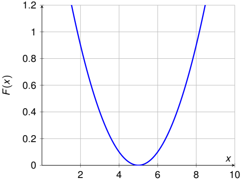


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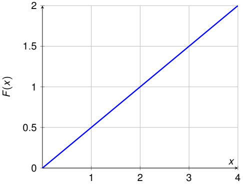


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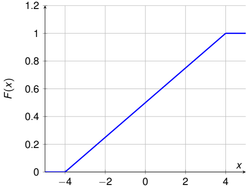


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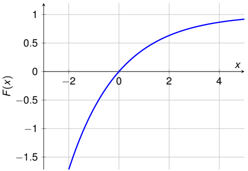


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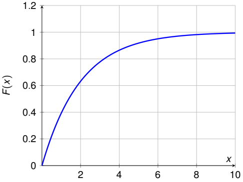


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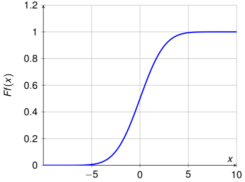
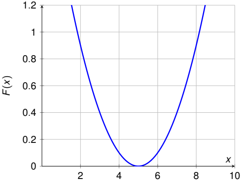
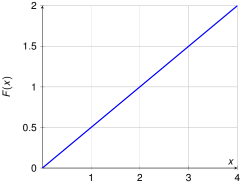



Yes.

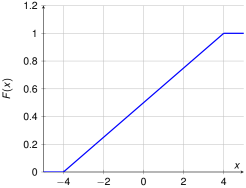
# Let’s play: cdf or not?



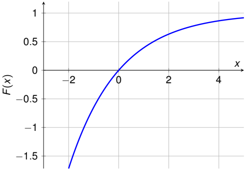

No. $F(x)$ has to be non-decreasing.



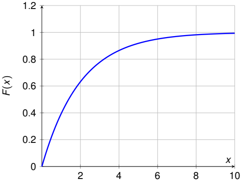

No. $F(x)$ has to be $\leq 1$.



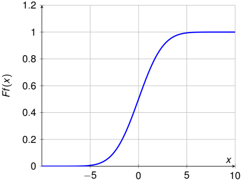

Yes.




No. $F(x)$ has to be $\geq 0$.




Yes.




Yes.

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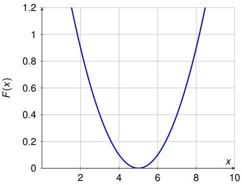
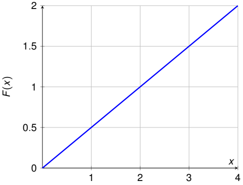
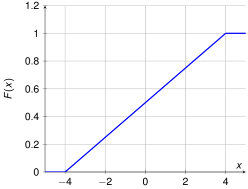
# Let’s play: cdf or not?



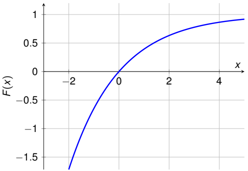

No. $F(x)$ has to be non-decreasing.



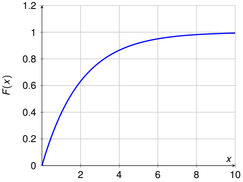

No. $F(x)$ has to be $\leq$ 1.



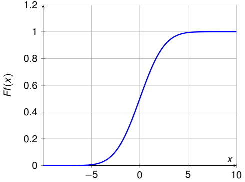

Yes.




No. $F(x)$ has to be $\geq$ 0.

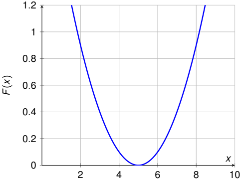



Yes.

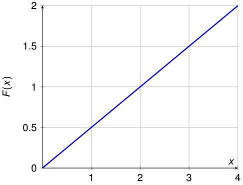



Yes.

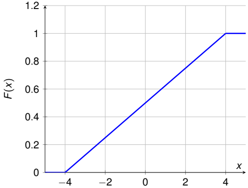
# Let's play: cdf or not?



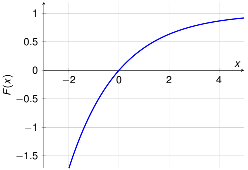

No. $F(x)$ has to be non-decreasing.



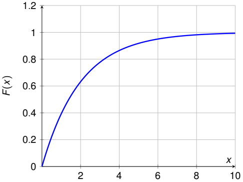

No. $F(x)$ has to be $\leq$ 1.



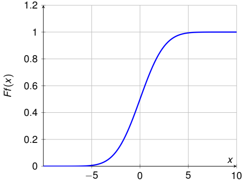

Yes.




No. $F(x)$ has to be $\geq$ 0.




Yes.




Yes.