Continuous random variables

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Lecture 7 b

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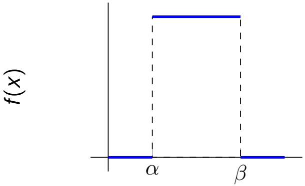
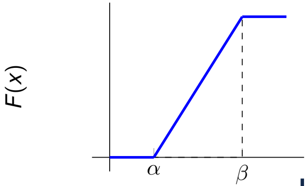
Uniform distributed random variables

- Not to be confused with its discrete counterpart.
- Here X is allowed to take *any value* in $[\alpha, \beta]$.

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha}, & \text{if } \alpha \leq x \leq \beta \\ 0, & \text{otherwise.} \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(y)dy = \begin{cases} 0, & \text{if } x < \alpha \\ \frac{x-\alpha}{\beta-\alpha}, & \text{if } \alpha \leq x \leq \beta \\ 1, & \text{if } x > \beta \end{cases}$$





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Exponentially distributed random variables

The second continuous random variable distribution we will see is immensely useful with a vast number of applications.

- Typically used to model time to next random event. Examples include:
  - time to next machine failure,
  - time to next accident,
  - time to next customer arrival,
  - time to next shark attack.
- Closely related to Poisson random variables and the geometric distribution (which are discrete).
- Similarly to Poisson random variables, we need a rate $\lambda > 0$.

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The exponential distribution

Given a parameter (called rate) $\lambda > 0$, the exponential distribution has:

- pdf: $f(x) = \begin{cases} \lambda \cdot e^{-\lambda x}, & \text{for } x \geq 0 \\ 0, & \text{for } x < 0 \end{cases}$
- cdf: $F(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{for } x \geq 0 \\ 0, & \text{for } x < 0 \end{cases}$

- **How is it related to the geometric distribution?**
  - In the discrete case, **the number of trials** to the first success was modeled as a geometric distribution.
  - In the continuous case, **the time** to the first success is modeled as an exponential distribution.

- **How is it related to the Poisson distribution?**
  - Time to next event: exponential distribution.
  - Number of events in some time: Poisson distribution.

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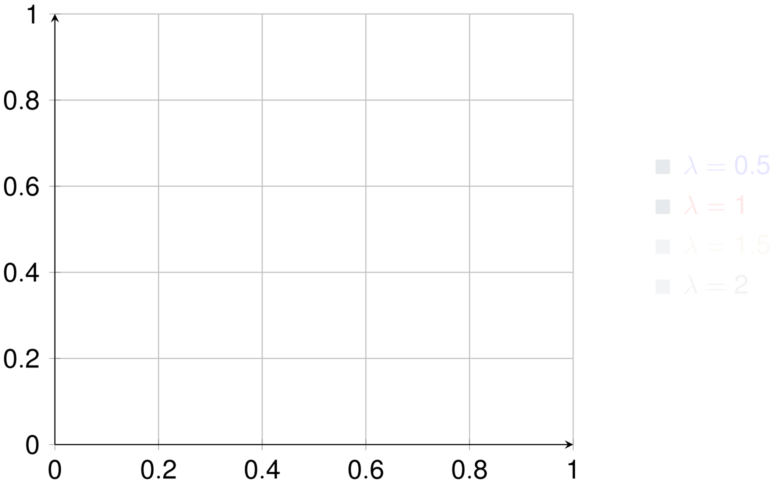
The exponential distribution pdf visually



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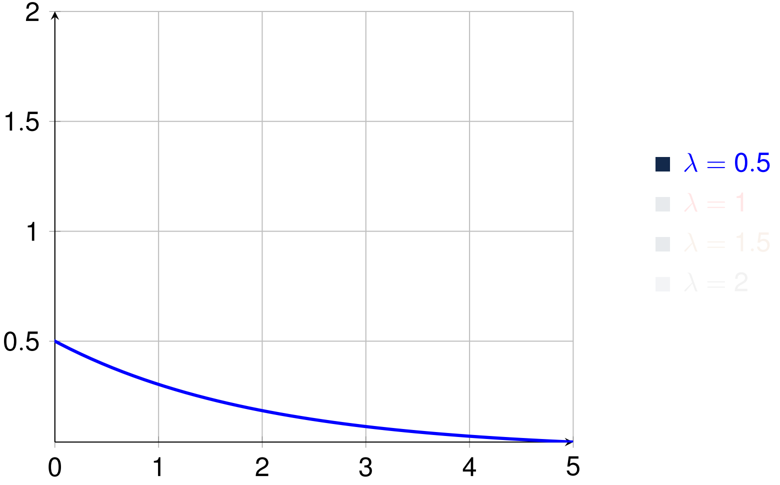

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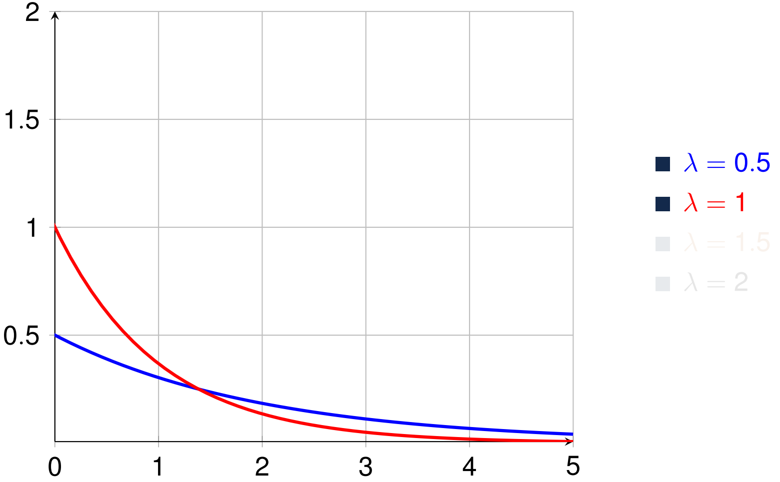

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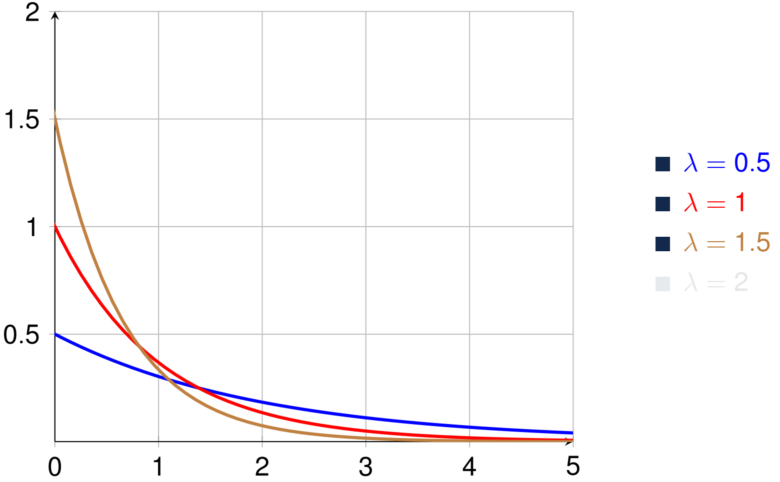

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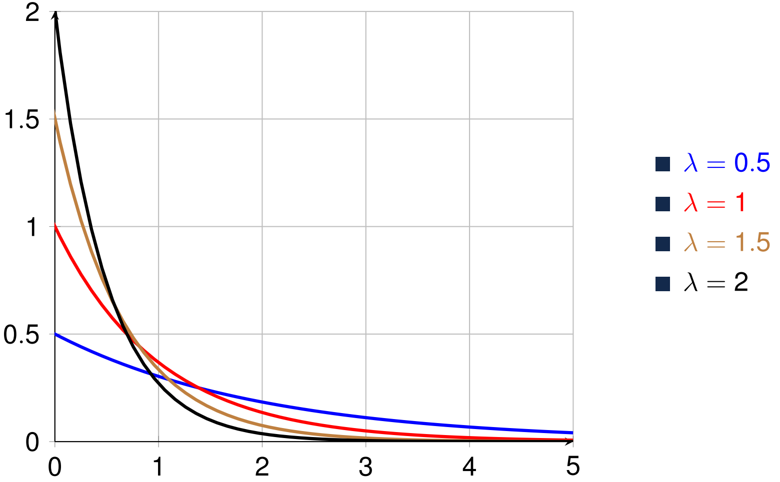

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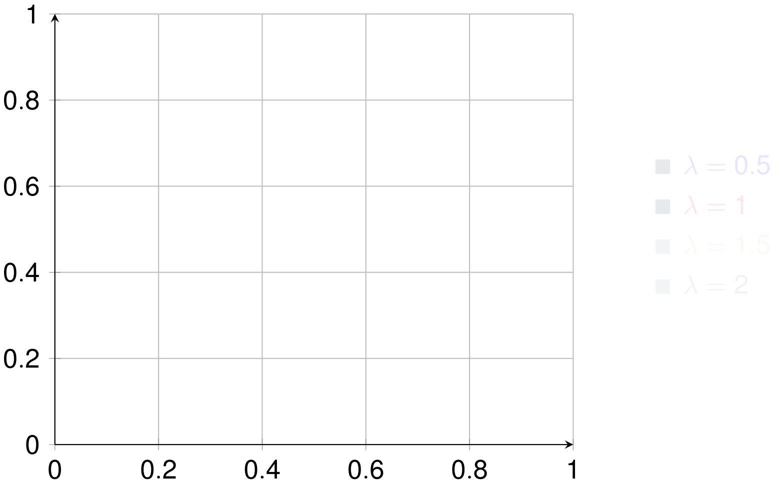

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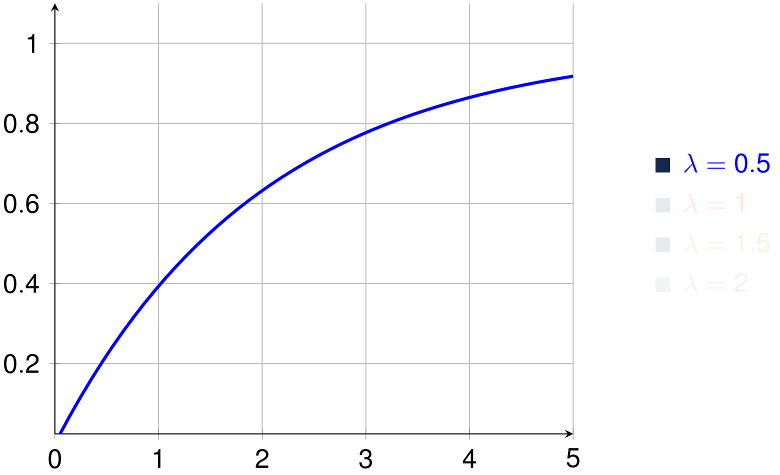

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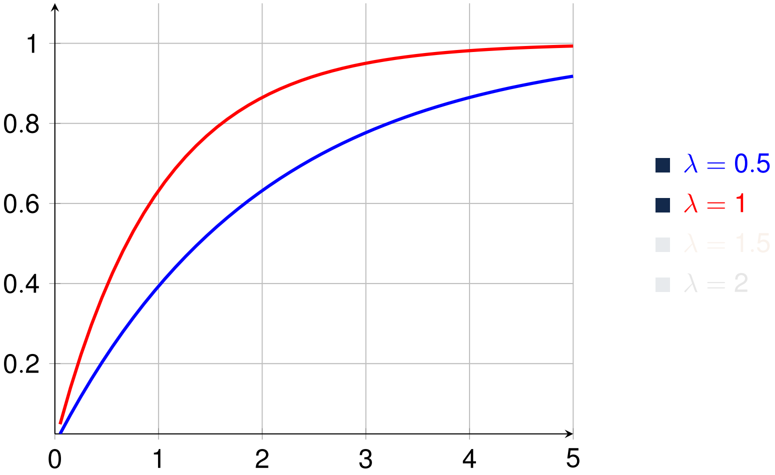

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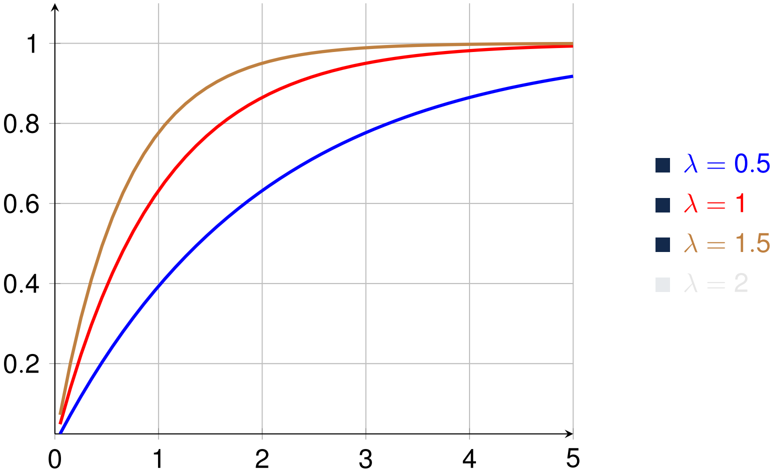

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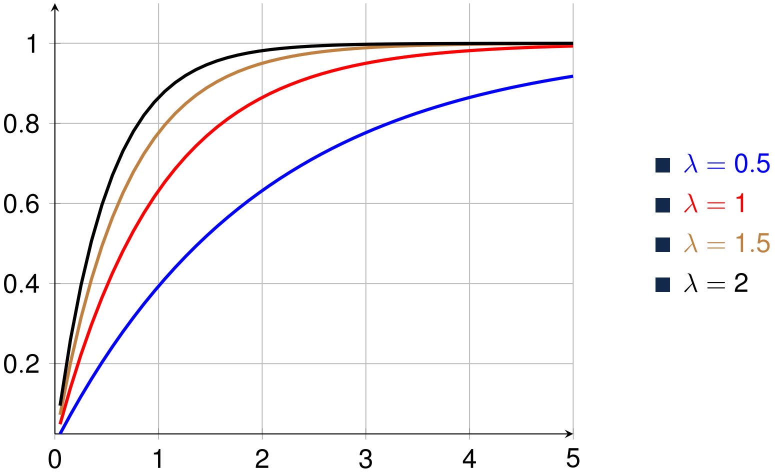

Expo. and Poisson distributed random variables

We said earlier that these random variables are siblings:

- Assume an event that happens in time that is **exponentially distributed with rate** λ.
- Then, the number of events that have taken place is **Poisson distributed with rate** λ.

Example

Historically, an emergency room after hours (10pm–6am) sees 48 requests every 8 hours. The time until the next patient arrives is exponentially distributed with that rate. What is the probability that the next patient arrives in the next 10 minutes? What is the probability there are 5 patients during the next hour?

Answer: Let's use $\lambda = \frac{48 \text{ requests}}{8 \text{ hours}} = 6$ requests per hour.

- The time to the next patient arriving is **exponentially** distributed.
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- Time to next patient:

$$P(T \leq 10 \text{ mins}) = P(T \leq 1/6 \text{ hrs}) = 1 - e^{-6 \cdot \frac{1}{6}} = 1 - e^{-1} = \\ = 0.6321.$$

- Number of patients:

$$P(X = 5) = e^{-6} \cdot \frac{6^5}{5!} = 0.1606.$$

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A car transmission fails in time that is exponentially distributed with a rate of 1 every 80,000 miles. What is the probability that the transmission does not fail within its first 40,000 miles? What is the probability that the transmission does not fail in the next 40,000 miles given that it has already been working for 80,000?

Answer: First, estimate the failure rate as $\lambda = 1/80000$.
$P(X > 40000) = 1 - F(40000) = 1 - (1 - e^{-\frac{40000}{80000}}) = e^{-0.5} = 0.607.$

We are looking for $P(X > 120000 | X > 80000)$. Let $s = 120000$ and $t = 80000$. Then:

$$P(X > s + t | X > s) =$$

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Memorylessness

Definition

A random variable X is said to be *memoryless* (without memory) if:

$$P(X > s + t | X > s) = P(X > t).$$

- Think of X as the lifetime of a printer. Then:
- the probability it still works after a year, and
- the probability it still works in Year 3, assuming it's survived until Year 2 are the same, if X is memoryless!

Exponentially distributed random variables are memoryless! They are the only continuous random variables with that property.

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