Continuous random variables

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Lecture 8

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ISE | Industrial & Enterprise Systems Engineering
GRAINGER COLLEGE OF ENGINEERING

Last time..

- Continuous random variables.
- Probability density functions (pdf, $f(x)$, relative likelihood) and cumulative distribution functions (cdf, $F(x) = P(X \leq x)$).
- Some key points include:
  - $F(x) = \int_{-\infty}^{x} f(y)dy$, which means that $f(x) = F'(x)$.
  - $P(X \in A) = \int_A f(x)dx$.
  - $P(\alpha \leq X \leq \beta) = F(\beta) - F(\alpha) = \int_{\alpha}^{\beta} f(x)dx$.
  - Recall that $P(X = \alpha) = 0$, for any continuous random variable X.
- We have seen two distributions so far:
  - Uniform;
  - Exponential.

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  - Uniform;
  - Exponential.

Today, we will discuss the Gamma and the Erlang distribution, and the ubiquitous normal distribution.

The Gamma distribution

We have the tools to answer two questions. Given a positive rate λ:

1. What is the probability that the next event happens during some time interval?
2. What is the probability that we see a number of events during some time interval?

Time to answer a third question:

3. What is the probability that the k-th event happens during some time interval?

Definition (The Gamma distribution)

A continuous random variable X defined over the interval of $[0, \infty)$ is Gamma distributed if it has probability density function given by

$$f(x) = \begin{cases} \frac{\lambda^k \cdot x^{k-1} \cdot e^{-\lambda x}}{\Gamma(k)}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0, \end{cases}$$

where $\lambda, k > 0$ are given parameters and $\Gamma(k)$ is the Gamma function.

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The Erlang distribution

- In this class, we will only deal with integer values of k, and hence

$$\Gamma(k) = (k-1)!$$

- We sometimes write that $X \sim Gamma(k, \lambda)$ if it follows the Gamma distribution with rate λ and shape parameter k.
- When parameter k is integer, we say the distribution is **Erlang**, and we write that $X \sim Erlang(k, \lambda)$.

Example

A customs officer at the border stops cars for random inspection; they stop every third vehicle – so they would stop the 3rd, the 6th, the 9th, and so on. Cars cross the border with a rate of 1 every 2 minutes. What is the probability they stop their first car within the first 10 minutes of work?

Answer: This is an Erlang distributed random variable with $k = 3$ and $\lambda = 1/2$ minutes: $f(x) = \frac{\left(\frac{1}{2}\right)^3 \cdot x^2 \cdot e^{-\frac{1}{2}x}}{\Gamma(3)} = \frac{1}{16}x^2 e^{-\frac{1}{2}x}$. To answer the question, we need

$$P(X \leq 10) = \int_0^{10} f(x)dx = \int_0^{10} \frac{1}{16}x^2 e^{-\frac{1}{2}x}dx = 0.87535.$$

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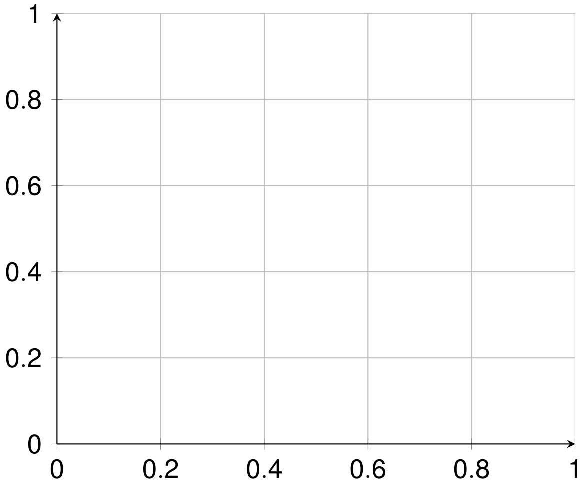
The Erlang distribution visually

Figure: The Erlang distribution probability density function visualized for different values of λ and integer values of k.



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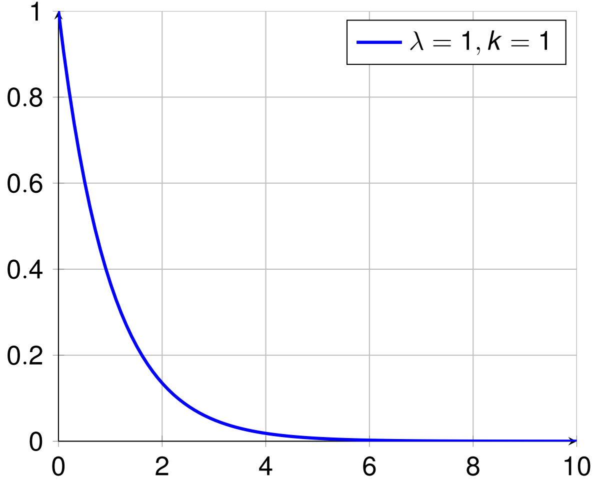

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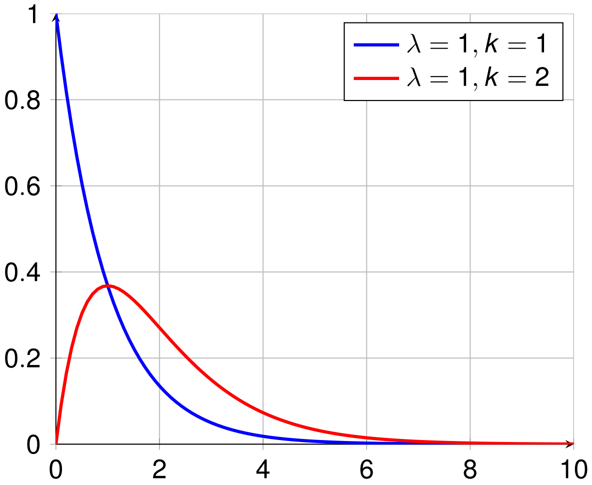

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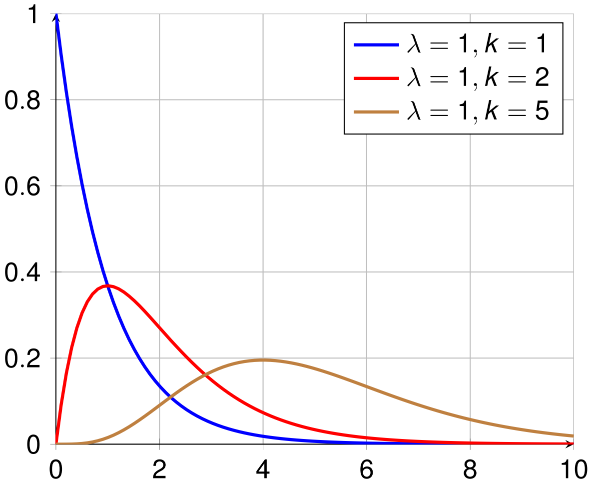

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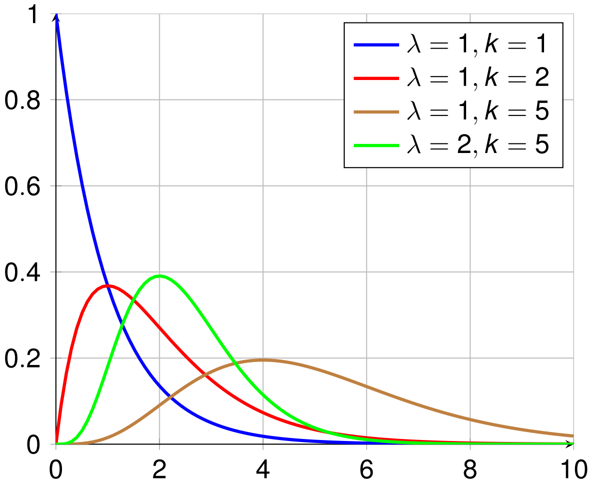

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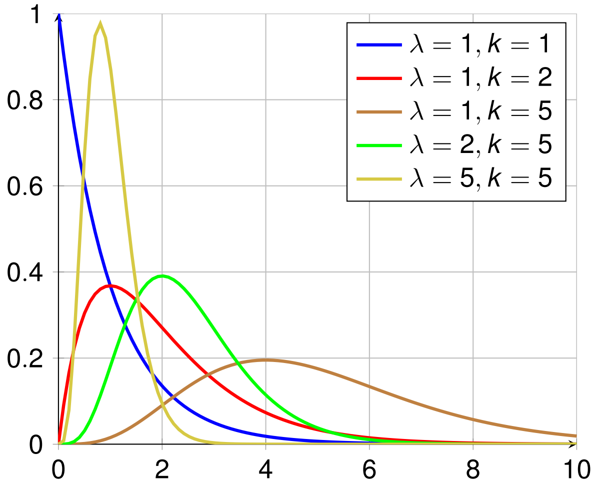

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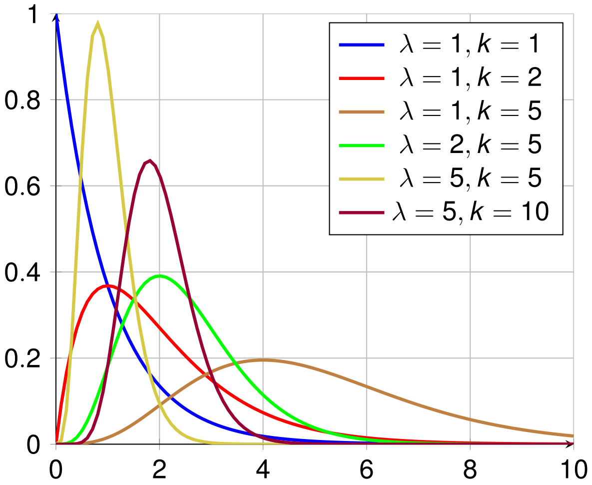

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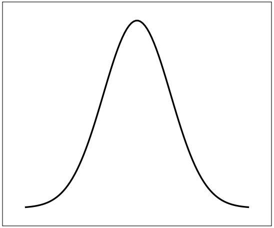
The normal distribution



- Probably the most well-studied continuous random variable.
- Given two parameters referred to as the mean (μ) and the standard deviation (σ) or variance (σ^2):
  - pdf: $f(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
  - cdf: $F(x) = \int_{-\infty}^{x} f(t) dt$
- We then say that a random variable X is $\mathcal{N}(\mu, \sigma^2)$ distributed.
- It is *symmetric* and *bell-shaped*.

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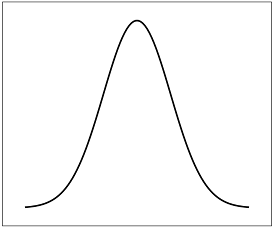

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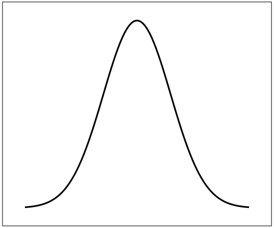

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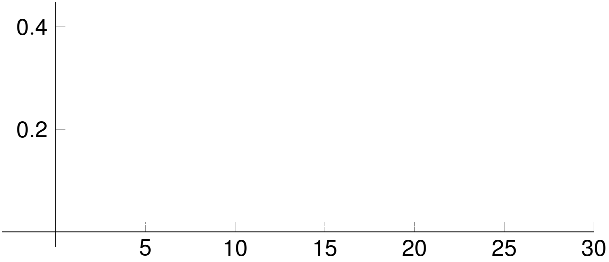
Mean and standard deviation/variance?

The *location* and the *spread* of the normal distribution are determined uniquely by its two defining parameters μ and σ (or σ^2):

Figure: Some examples of how the normal distribution is affected by its mean and standard deviation.



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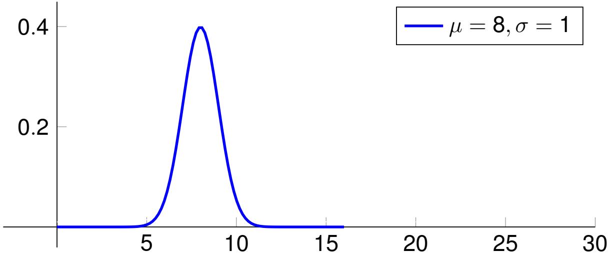

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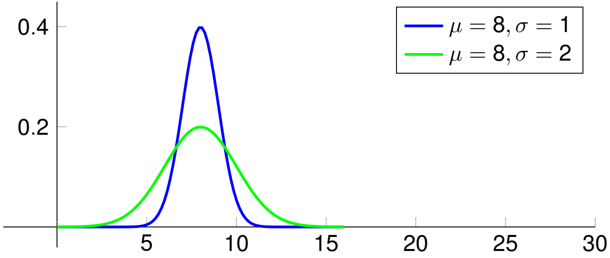

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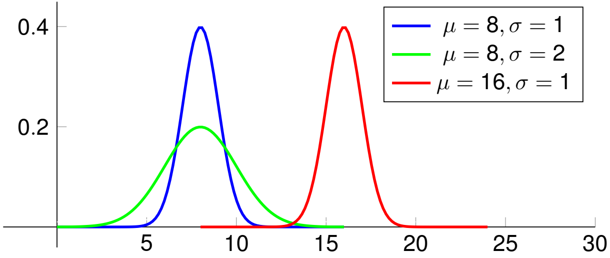
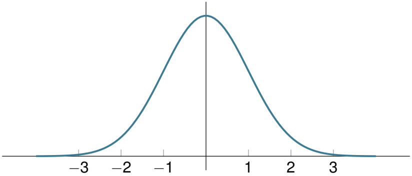


Standard normal distribution



- Due to its applicability, we have also devised a **standard** version.
- In this version, we have that $\mu = 0, \sigma = 1$.
- If a random variable is following a standard normal distribution, we write that it is $\mathcal{N}(0, 1)$ distributed.
- We have tables that contain useful values for the cdf (so that we do not have to calculate an integral every time).
- Any normal distribution can be converted to the standard normal distribution through a procedure usually called a *z-transform*.

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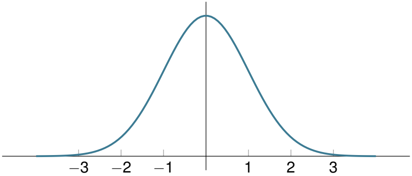

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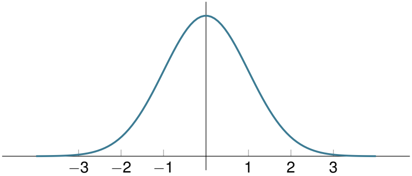

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z-transforms and the z-table

Remark

If X is $\mathcal{N}(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma}$ is $\mathcal{N}(0,1)$.

- Note: we usually write $\boxed{f(x), F(x)}$ to signal the pdf and cdf of a non-standard normal distribution, and $\boxed{\phi(z), \Phi(z)}$ to signal the pdf and cdf of the standard normal distribution.
- The z-transform equivalency implies that we may:
  1. calculate the corresponding z value through the transformationl
  2. then use a table containing probabilities to find that z value;
  3. and hence obtain the probability we are after.

Let X be $\mathcal{N}(400, 400)$ (i.e., $\sigma^2 = 400 \implies \sigma = 20$).

- Example 1: $P(X \leq 400)$?
  - $Z = \frac{X-\mu}{\sigma} = \frac{400-400}{20} = 0.$
- Example 2: $P(X \leq 451)$?
- Example 3: $P(X \leq 375)$?

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  1. calculate the corresponding z value through the transformationl
  2. then use a table containing probabilities to find that z value;
  3. and hence obtain the probability we are after.

Let X be $\mathcal{N}(400, 400)$ (i.e., $\sigma^2 = 400 \implies \sigma = 20$).

- Example 1: $P(X \le 400)$?
  - $z = \frac{x-\mu}{\sigma} = \frac{400-400}{20} = 0.$
  - $P(X \le 400) = P(Z \le 0).$
- Example 2: $P(X \le 451)$?
  - $z = \frac{x-\mu}{\sigma} = \frac{451-400}{20} = 2.55.$
  - $P(X \le 451) = P(Z \le 2.55).$
- Example 3: $P(X \le 375)$?
  - $z = \frac{x-\mu}{\sigma} = \frac{375-400}{20} = -1.25.$
  - $P(X \le 375) = P(Z \le -1.25).$

z-transforms and the z-table

NORMAL CUMULATIVE DISTRIBUTION FUNCTION ($\Phi(z)$)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Example solutions

- Example 1: $P(X \leq 400) = P(Z \leq 0) = \Phi(0)$.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	**0.5000**	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359

And, hence, $P(X \leq 400) = 0.5$.

- Example 2: $P(X \leq 451) = P(Z \leq 2.55) = \Phi(2.55)$.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	**0.9946**	0.9948	0.9949	0.9951	0.9952

Which implies that $P(X \leq 451) = 0.9946$.

- Example 3: $P(X \leq 375) = P(Z \leq -1.25) = \Phi(-1.25)$.
  - Many z tables do not have negative values.
  - Thankfully, due to the symmetry of the normal distribution, we may use the positive value!

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	**0.8944**	0.8962	0.8980	0.8997	0.9015

Finally, we have that $\Phi(-z) = 1 - \Phi(z) = 1 - 0.8944 = 0.1056$.

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