Variances

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Lecture 9b

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GRAINGER COLLEGE OF ENGINEERING

Variance

The second interesting quantity for a random variable is its *variance*. It is defined as:

$$Var[X] = E\left[(X - E[X])^2\right]$$

Specifically, we have:

- for discrete random variables:

$$Var[X] = \sum_{x \in S} (x - E[X])^2 \cdot p(x)$$

- for continuous random variables:

$$Var[X] = \int_{-\infty}^{+\infty} (x - E[X])^2 \cdot f(x)dx$$

In both cases, it can be shown that:

$$Var[X] = E\left[X^2\right] - (E[X])^2$$

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Variance examples

Example

What is the variance of random variable X when it represents the side of a "fair" die?

Answer: We first calculate

$$E[X] = \sum_{i=1}^{6} i \cdot p(i) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{21}{6} = 3.5.$$

We can now use it to calculate the variance:

$$Var[X] = \sum_{i=1}^{6} (i - E[X])^2 \cdot p(i) = (1 - 3.5)^2 \cdot \frac{1}{6} + (2 - 3.5)^2 \cdot \frac{1}{6} + (3 - 3.5)^2 \cdot \frac{1}{6} + (4 - 3.5)^2 \cdot \frac{1}{6} + (5 - 3.5)^2 \cdot \frac{1}{6} + (6 - 3.5)^2 \cdot \frac{1}{6} = \frac{35}{12}$$

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Variance examples

Example

Let X be a continuous random variable measuring the current (in milliamperes, mA) in a wire with pdf $f(x) = 0.05$, for $0 \leq x \leq 20$. What is the variance of X?

Answer: We again calculate

$E[X] = \int_0^{20} x \cdot f(x) = \int_0^{20} 0.05x = \frac{0.05x^2}{2}\Big|_0^{20} = 10$. Once more:

$Var[X] = \int_0^{20} (x - E[X])^2 \cdot f(x)dx = \int_0^{20} (x-10)^2 \cdot f(x)dx =$

$\frac{0.05 \cdot (x-10)^3}{3}\Big|_0^{20} = 16.67.$

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Variances of discrete random variables

- Bernoulli with probability p (assume failure=0 & success=1):

$$Var[X] = (1-p)^2 \cdot p + (0-p)^2 \cdot (1-p) = p \cdot (1-p).$$

- Binomial with parameters p and n:

$$Var[X] = n \cdot p \cdot (1-p).$$

- Geometric with parameter p:

$$Var[X] = \frac{1-p}{p^2}.$$

- Poisson with parameter λ:

$$Var[X] = \lambda.$$

much like its mean!

- Hypergeometric with parameters N, K, n:

$$Var[X] = n \cdot \frac{K}{N} \cdot \frac{N-K}{N} \cdot \frac{N-n}{N-1}.$$

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Variances of continuous random variables

- Uniform between α and β:

$$Var[X] = \frac{1}{12}(\beta - \alpha)^2.$$

Example

If the next bus arrives uniformly in the next 10 minutes, then the next bus arrival has a variance of $Var[X] = \frac{100}{12} = 8.33$ *minutes*^{2}.

- Normal with parameters μ, σ^2:

$$Var[X] = \sigma^2.$$

Example

If grades are normally distributed with $\mathcal{N}(80, 12)$, *then the variance of a student grade in the class is* $Var[X] = 12$.

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- Exponential with rate λ:

$$Var[X] = \frac{1}{\lambda^2}.$$

Example

If cars pass through an intersection with rate $\lambda = 2$ per minute, then the variance of the next car arrival is $Var[X] = \frac{1}{\lambda^2} = 0.25$ minutes^2.

- Gamma/Erlang with parameters λ and k:

$$Var[X] = \frac{k}{\lambda^2}.$$

Example

If cars pass through an intersection with rate $\lambda = 2$ per minute, then the variance of the $k = 30$-th car arrival is $Var[X] = \frac{30}{\lambda^2} = 7.5$ minute^2.

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Properties

Let α, β be real numbers, and X, Y random variables. Then:

1. $Var[X] \geq 0$.
2. $Var[\alpha] = 0$.
    - The above two properties lead to $Var[X + \alpha] = Var[X]$.
3. $Var[\alpha \cdot X + \beta] = \alpha^2 Var[X]$
4. $Var[X + Y] = Var[X] + Var[Y]$.
    - When X and Y are **independent random variables**.
    - Generalizes to $Var\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} Var[X_i]$.
    - Can be generalized even further to:

$$Var\left[\sum_{i=1}^{n} (\alpha_i \cdot X_i + \beta_i)\right] = \sum_{i=1}^{n} \alpha_i^2 \cdot Var[X_i].$$

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1. μ or $E[X]$ called the mean or expectation (or expected value) of a random variable X.
2. σ^2 or $Var[X]$ called the variance of a random variable X.
3. σ or $SD[X]$ called the standard deviation of a random variable X.

   By definition $Var[X] = SD[X]^2$.

4. We also saw how to calculate the expectation and variance of many known probability distributions.
5. We finally discussed several expectation and variance properties.

Note: the mean and the variance alone do not identify the distribution!

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