Lecture 1 Worksheet

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Every worksheet will work as follows.

1. You will be asked to form a group with other students in the class: you can make this as big or as small as you'd like, but groups of 4-5 work best.
2. Read through the worksheet, discussing any questions with the other members of your group.
   - You can call me at any time for help!
   - I will also be interrupting you for general guidance and announcements at random points during the class time.
3. Answer each question (preferably in the order provided) to the best of your knowledge.
4. While collaboration between students is highly encouraged and expected, each student has to submit their own version.
5. You will have 24 hours (see gradescope) to submit your work.

Activity 1: Playing with coins

Problem 1

A coin is tossed 4 times in a row and we mark whether it has come up Heads or Tails each time. Explain (in a sentence) why this is a random experiment. [1]

Answer to Problem 1.

[1] Consider what happens if we repeat this process. Do we always get the same sequence?

Problem 2

In the experiment from Problem 1, what is the sample space? [2] What is the cardinality of the sample space?

Answer to Problem 2.

$S =$

$|S| =$

[2] In general, there are multiple ways to define a sample space. Here, we may want to focus on each individual coin toss as it happens. For example, $\{Heads, Heads, Tails, Heads\}$ could be a potential outcome, whereas $\{Tails, Tails, Tails, Heads\}$ would be another.

Problem 3

Once again, consider the experiment from Problem 1. What is the cardinality of the following events:

1. Get the sequence Heads, Tails, Tails, Heads.
2. The third coin toss comes up Heads.
3. Get at least three Heads.

Answer to Problem 3.

1.

2.

3.

You know what's interesting? When all outcomes in the sample space are equally likely, then the "likelihood" of an event happening (*probability*) is equal to the cardinality of the event over the cardinality of the sample space! For example, "Get at least three Heads" has cardinality 5, and the sample space S has cardinality $|S| = 16$ for a "likelihood" of $5/16 = 31.25\%$. [3]

[3] More on that next time!

Problem 4

Consider the events in Problem 3. Are the first and the second events mutually exclusive? How about the first and the third events? Finally, what can you say about the second and the third events? Justify (in a sentence) your answer.

Answer to Problem 4.

1. "Get the sequence Heads, Tails, Tails, Heads" and "The third coin toss comes up Heads":

2. "Get the sequence Heads, Tails, Tails, Heads" and "Get at least three Heads":

3. "The third coin toss comes up Heads" and "Get at least three Heads":

Activity 2: An experimental design

Problem 5

An experiment happens in an environment where the temperature is always between 20 and 100 Fahrenheit – that is, the temperature belongs to $S = [20, 100]$. Define the events $A = [80, 100]$ (temperature is greater than or equal to 80 F), $B = [20, 40]$ and $C = [32, 85]$ (that is, the temperature is between 32 and 85 Fahrenheit). We say the experiment is successful when the temperature belongs to C, and we say event C has happened. In other cases, the experiment is unsuccessful. We also say that the experiment is hot when the temperature belongs to A, and we say that event A has happened. Similarly, we say that the experiment is cold when the temperature belongs to B, and we say that B has happened. [4]

[4] Be careful with the inclusion and exclusion notation [,] and (,), respectively.

Answer to Problem 5.

1. What is $\overline{A}$ mathematically?

2. What is $B \setminus A$ mathematically?

3. What is $B \cap C$ mathematically?

4. What is $(\overline{A} \cap B) \cup (\overline{A} \cap C)$ mathematically?

5. Define the set of outcomes where an experiment that is both successful and in regular temperatures (i.e., neither hot nor cold) in set notation and mathematically.

Activity 3: Set and cardinality properties

We saw many interesting properties in the first pre-lecture video and the accompanying slides. Now, it is time to see their derivations. To begin with, consider the first of the two DeMorgan's law we saw in the lecture notes:

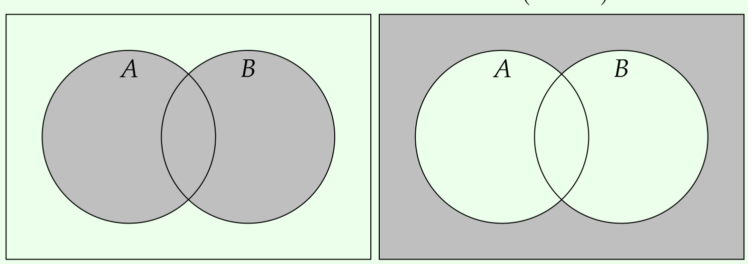
$$\boxed{\overline{(A \cup B)} = \overline{A} \cap \overline{B}.}$$

Problem 6

In the two Venn diagrams below, we mark in gray the event $(A \cup B)$ and $\overline{(A \cup B)}$ (corresponding to the left hand side). We have solved this for you, so you can move to Problem 7.

Answer to Problem 6.

$A \cup B$ $\qquad\qquad$ $\overline{(A \cup B)}$

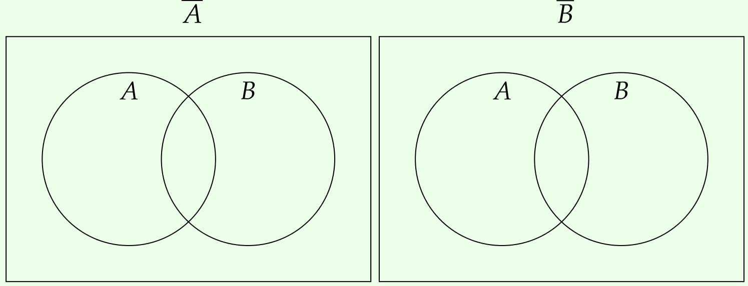


Problem 7

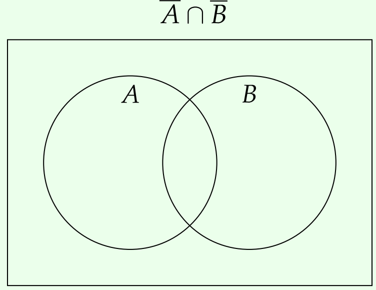
In the three Venn diagrams provided, mark the events $\overline{A}$, $\overline{B}$, and $\overline{A} \cap \overline{B}$ (corresponding to the right hand side of the DeMorgan's law).

Answer to Problem 7.

$\overline{A}$ $\qquad\qquad$ $\overline{B}$



$\overline{A} \cap \overline{B}$



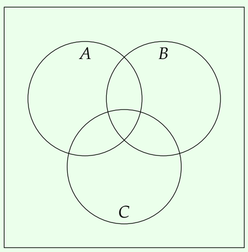
Note how the last Venn diagram in Problem 7 (the one for $\overline{A} \cap \overline{B}$) is the same as the last Venn diagram in Problem 6 (the one corresponding to $\overline{(A \cup B)}$, "showing" DeMorgan's law.

Problem 8

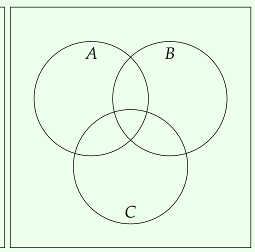
Based on the previous constructive derivation you did, what can you say about the extension of this DeMorgan's law to more than 2 events? Specifically, what is $\overline{(A \cup B \cup C)}$? Use the diagrams below to do something similar to what you did in Problems 6 and 7.

Answer to Problem 8.

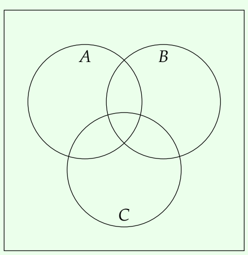
$A \cup B \cup C$



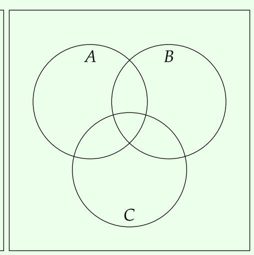
$\overline{(A \cup B \cup C)}$



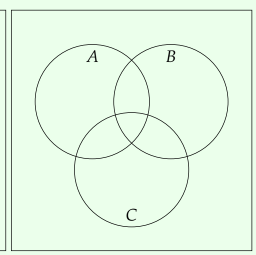
$\overline{A}$



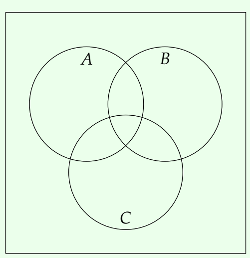
$\overline{B}$



$\overline{C}$



$\overline{A} \cap \overline{B} \cap \overline{C}$



Finally, we have that:

$$\overline{(A \cup B \cup C)} =$$

Problem 9

Consider two **mutually exclusive** [5] events A and B. What can you say about the cardinality of $A \cap B$ and of $A \cup B$? You may express your answer as a function of the individual cardinalities $|A|$ and $|B|$.

[5] That is, two events that *share no common outcomes*.

Answer to Problem 9.

- $|A \cap B| =$
- $|A \cup B| =$

Problem 10

It is true that for two general sets A, B, we have that $|A \cup B| = |A| + |B| - |A \cap B|$. Let us construct a proof for the statement using your answers in Problem 9.

Consider an event X that is comprised of **three mutually exclusive events** X_1, X_2, X_3. What can you say about the cardinality of X and the cardinalities of X_1, X_2, X_3? Circle the (one) correct answer.

Answer to Problem 10.

$$|X| \quad \begin{matrix} < \\ \leq \\ = \\ \geq \\ > \end{matrix} \quad |X_1| + |X_2| + |X_3|$$

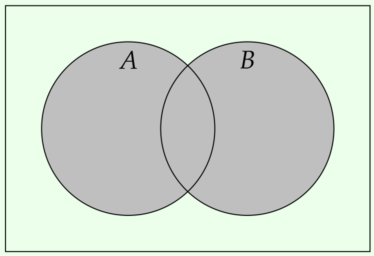
This is true, in general, no matter how many mutually exclusive events we can break an event down to. Say that event X consisted of n mutually exclusive events $X_1, X_2, \ldots, X_n$, we could write that its cardinality $|X|$ is equal to the summation of the individual event cardinalities:

$$|X| = \sum_{i=1}^{n} |X_i|.$$

Problem 11

Based on your observation in Problem 10, can we think of $A \cup B$ as three mutually exclusive events? Check the following Venn diagram and mark the **three mutually exclusive events** that describe the greyed area. How are they described mathematically?

Answer to Problem 11.



Problem 12

Combine your observations from Problems 10 and 11 to derive that $|A \cup B| = |A| + |B| - |A \cap B|$.

Answer to Problem 12.