Lecture 12 Worksheet

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Every worksheet will work as follows.

1. You will be asked to form a group with other students in the class: you can make this as big or as small as you'd like, but groups of 4-5 work best.
2. Read through the worksheet, discussing any questions with the other members of your group.
   - You can call me at any time for help!
   - I will also be interrupting you for general guidance and announcements at random points during the class time.
3. Answer each question (preferably in the order provided) to the best of your knowledge.
4. While collaboration between students is highly encouraged and expected, each student has to submit their own version.
5. You will have 24 hours (see gradescope) to submit your work.

Activity 1: Jointly distributed discrete random variables

During our last worksheet, we saw two discrete random variables $X = \{1, 2, 3\}$ and $Y = \{10, 20\}$ that were jointly distributed with probability mass function:

$$f_{XY}(x, y) = \frac{20}{27} \cdot \frac{x+1}{y}.$$

Additionally, we can calculate the marginal distribution of X and Y:

$$f_X(x) = \sum_y f_{XY}(x, y) = \frac{20}{27}\frac{x+1}{10} + \frac{20}{27}\frac{x+1}{20} = \frac{x+1}{9},$$

$$f_Y(y) = \sum_x f_{XY}(x, y) = \frac{20}{27}\frac{2}{y} + \frac{20}{27}\frac{3}{y} + \frac{20}{27}\frac{4}{y} = \frac{180}{27y} = \frac{60}{9y}.$$

Finally, recall that the conditional distribution for X given $Y = y$ can be calculated as:

$$f_{X|Y=y}(x) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{\frac{20}{27} \cdot \frac{x+1}{y}}{\frac{60}{9y}} = \frac{x+1}{9}.$$

It seems that the conditional and the marginal distribution are the same – this is not always the case! This only happens when X and Y are independent. More on that later today.

Problem 1: Expectations and variances

What is the expectation of X and what is the variance of X? [1]

Answer to Problem 1.

[1] Recall that the expectation of one of two jointly distributed random variables can be found by properly summing (if discrete, as is the case here) or integrating (when continuous) its **marginal distribution**. The marginal distribution is given in Page 1.

Problem 2: Conditional expectations and variances

What is the expectation of X and what is the variance of X *given that* $Y = 10$? [2]

Answer to Problem 2.

[2] The previous hint still applies! However, we now replace the marginal with the **conditional distribution**. Again, the conditional distribution is given in Page 1.

Problem 3: Independent? Covariance? Correlation?

As hinted at when we calculated the marginal and conditional distributions, it appears that $f_X(x) = f_{X|Y=y}(x)$. Equivalently, we could make the observation that $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$. Hence, the two random variables X and Y are independent!

Based on your this observation of independence, what is the covariance? What is the correlation? [3]

[3] Recall that two independent random variables have **zero** covariance and, consequently, **no** correlation.

Answer to Problem 3.

$\sigma_{XY} = Cov\left[X, Y\right] =$

$\rho_{XY} = Corr\left[X, Y\right] =$

Activity 2: Jointly distributed continuous random variables

Consider two jointly distributed *continuous* random variables X, Y with domains $0 \leq X \leq 2$ and $0 \leq Y \leq 1$ and with joint probability density function equal to:

$$f_{XY}(x,y) = \frac{3}{4}x^3y^2.$$

Problem 4: Warm-up with marginal distributions

Let's repeat what we had done during our previous lecture. What are the marginal distributions of X and Y? [4]

[4] As a reminder, the marginal distribution of X will be a function of x and the marginal distribution of Y will be a function of y.

Answer to Problem 4.

$f_X(x) = \int_0^1 f_{XY}(x,y)dy =$

$f_Y(y) = \int_0^2 f_{XY}(x,y)dx =$

Problem 5: Getting the expectations

What are the expectations of X and Y? Don't forget that they are defined over *different domains*! [5]

[5] X is defined over $[0, 2]$, whereas Y is defined over the range $[0, 1]$.

Answer to Problem 5.

Problem 6: Independent?

Are X and Y independent? Why/Why not? [6]

[6] Check whether $f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$: if so, then they are independent!

Answer to Problem 6.

Activity 3: When X and Y restrict each other

Assume that random variables X and Y are jointly distributed with probability density function $f_{XY}(x, y) = \frac{1}{4}(x + y)$ defined over $0 \leq X \leq Y \leq 2$. Note how random variable X always takes values that are at most as big as the value of random variable Y. [7] X and Y are not independent; this is clear from their definition, as knowing the one restricts the values the other one may take. What is the covariance of X and Y then? Well, to answer that we will need a lot of things. Namely, we need:

1. the marginal distributions $f_X(x), f_Y(y)$;
2. the expectation $E[X], E[Y]$;
3. the expectation of function $X \cdot Y$: $E[X \cdot Y]$;
4. finally, you'll get

$$Cov[X, Y] = E[X \cdot Y] - E[X] \cdot E[Y].$$

Let's get to it!

[7] If you are wondering how this is a valid pdf, we may show that the double integration is equal to 1. Be **very careful** with how you are integrating this. Following are the two correct ways (for an incorrect way, look at the notes!):

$$\int_0^2 \int_0^y \frac{1}{4}(x + y)\, dxdy = 1$$

$$\int_0^2 \int_x^2 \frac{1}{4}(x + y)\, dydx = 1$$

Problem 7: Marginal distributions

What are the marginal distributions of X and Y? Verify that X and Y are not independent by checking whether $f_X(x) \cdot f_Y(y)$ is different than $f_{XY}(x, y)$. Here, we help you get $f_X(x)$ started as a hint.

Answer to Problem 7.

$$f_X(x) = \int_x^2 f_{XY}(x, y)dy =$$

Problem 8: Expectation of X and Y

Using the marginal distributions from earlier, what is the expectation of X and what is the expectation of Y? [8]

Answer to Problem 8.

[8] Ok.. So expectations are typically found by integrating the marginal over the domain of the random variable. What is the domain here? Well, if Y is "gone", the domain for random variable X is $[0, 2]$, and if X is gone the domain of Y is also $[0, 2]$...

Problem 9: Expectation of $X \cdot Y$

And.. what is the expectation of $X \cdot Y$? [9]

Answer to Problem 9.

[9] Assume we have two random variables X and Y that are jointly distributed with joint pdf $f_{XY}(x, y)$. Then, for a function of those random variables, $g(X, Y)$, its expectation is $\iint g(x, y) f_{XY}(x, y) dxdy$. But what about the integral limits? See Hint 7 in Page 5 :)

Problem 10: Covariance and correlation

Almost there. So, what is the covariance and what is the correlation of X and Y? For the correlation calculations as well as for any other remaining parts in this specific activity, use that $Var[X] = \frac{43}{180}$ and $Var[Y] = \frac{3}{20}$.

Answer to Problem 10.

Problem 11: Applying the variance identity

In the lecture notes, we discuss how the variance identity becomes [10]:

$$Var[aX + bY] = a^2 Var[X] + b^2 Var[Y] + 2abCov[X, Y].$$

So, assume we are interested in the variance of random variable $Z = 3 \cdot X + 2 \cdot Y$: what can it be evaluated as?

Answer to Problem 11.

[10] Recall that if X and Y are independent, then $Cov[X, Y] = 0$, and $Var[aX + bY] = a^2 Var[X] + b^2 Var[Y]$.