Lecture 17 Worksheet

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Every worksheet will work as follows.

1. You will be entered into a Zoom breakout session with other students in the class.
2. Read through the worksheet, discussing any questions with the other participants in your breakout session.
   - You can call me using the "Ask for help" button.
   - Keep in mind that I will be going through all rooms during the session so it might take me a while to get to you.
3. Answer each question (preferably in the order provided) to the best of your knowledge.
4. While collaboration between students in a breakout session is highly encouraged and expected, each student has to submit their own version.
5. You will have 24 hours (see Compass) to submit your work.

Worksheet 1: Streaming services and their data

A TV streaming service has collected a lot of information from several customers and is interested in analyzing it for patterns about their streaming habits. More specifically, they have collected data on two items: how much time they have spent using the service (from login to closing the tab/exiting the app) and how many episodes they have watched. Part of the data follows.

	Session #									
	1	2	3	4	5	6	7	8	9	10
Time (in hours)	0.75	1.20	1.33	0.97	0.80	1.43	0.87	1.41	1.09	1.05
# of episodes	1	3	3	3	2	5	1	5	3	1

Let us help them find good estimators using the method of moments! As a reminder, for the method of moments, we:

1. calculate (generally, there are exceptions) as many population and sample moments as the unknown parameters.
2. equate each population moment with the corresponding sample moment.
3. solve a system of equations.

Problem 1: The method of moments for a uniform distribution

The streaming service assumes that people spend time that is **uniformly distributed** in $[\theta, 2\theta]$ using their service. Use the method of moments to provide an estimator for θ, $\hat{\Theta}$. What is the point estimate $\hat{\theta}$ you get when using the data of the previous table? [1]

[1] Recall again that for the uniform distribution (continuous) the pdf is given as $f(x) = \frac{1}{b-a}$ when $x \in [a, b]$.

Answer to Problem 1.

Problem 2: The general uniform distribution

What if the streaming time is not uniformly distributed in $[\theta, 2\theta]$ but is instead more generally distributed in $[a, b]$? How would we go about estimating a and b? What are the point estimates $\hat{a}$ and $\hat{b}$ when you use the data from the earlier table?

Answer to Problem 2.

Problem 3: Number of episodes as a Poisson distributed random variable

Let us turn our focus to the number of episodes users watch. We assume the number of episodes watched during a session is a Poisson distributed random variable with rate λ; alas, λ is not known.

Using the method of moments, provide an estimator for λ. Use that estimator on the data from the table to obtain a point estimate $\hat{\lambda}$.

Answer to Problem 3.

Worksheet 2: Coming up with an estimator

The outcome of an experiment is a number between 0 and 1 (where 0 is considered an utter failure and 1 is considered a big success). The outcome is distributed with pdf $f(x) = \theta\left(x - \frac{1}{2}\right) + 1$, $0 \leq x \leq 1$, where θ is an unknown parameter.

Problem 4: Applying the method of moments

Using the method of moments, obtain an estimator for θ.

Answer to Problem 4.

Hopefully, you have gotten that $\hat{\Theta} = 12\overline{X} - 6$, where $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the average of n observations.
If you did not get that result, go back, find the mistake, and correct it. Did you take the first population and the first sample moments and equate them? Did you remember to equate the first sample moment (the sample average $\overline{X}$) to the first population moment (expected value $E[X] = \int_0^1 xf(x)dx = \frac{\theta+6}{12}$)?

Problem 5: MSE

What is the MSE of your estimator? [2] If you get stuck remember you will need that $E\left[\overline{X}\right] = E[X]$ and $Var\left[\overline{X}\right] = \frac{1}{n}Var[X]$. [3]

Answer to Problem 5.

[2] As a reminder, for an estimator $\hat{\Theta}$, the mean square error is:

$$MSE = bias\left[\Theta\right]^2 + Var\left[\hat{\Theta}\right].$$

[3] Additionally, remember that $Var[X] = \int_0^1 x^2 f(x)dx - (E[X])^2$...

Problem 6: Application

For this experiment, we have collected a sample of 6 items and found them equal to $X_1 = 0.8, X_2 = 0.83, X_3 = 0.95, X_4 = 0.72, X_5 = 0.85, X_6 = 0.65$. What is a good point estimate for θ? Use the point estimator you came up with in Problem 5. Using that estimate, what is the probability the experiment is at least a moderate success? Assume that moderate success is any value above 0.75.

Answer to Problem 6.

Worksheet 3: The Pareto distribution

Problem 7: A 2×2 system

The Pareto distribution (named after Italian mathematician and engineer, Vlifredo Pareto) is described by $f(x) = \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}}$ for $x \geq \beta$. Originally it was designed to apply to the distribution of wealth where a small portion of the population accounts for a large part of the wealth; but it has been successfully applied in other settings, too. In this exercise, you are asked to estimate α and β using the method of moments. You may assume that $\alpha > 2$ and $\beta \geq 1$. [4]

[4] No need to solve the system of equations. Simply set it up as if you had a sample of n observations $X_1, X_2, \ldots, X_n$.

Answer to Problem 7.

Worksheet 4: Special case

In this last activity, we have you run into a weird case. What if the first moment does not work? [5] Then, we need to take the second moments and equate them. Here is such a case.

[5] Maybe the first moments do not exist, or are not a function of the parameter.

Problem 8: A weird situation

Let X be a continuous population distributed with probability density function $f(x) = \frac{1}{2\beta}e^{-\frac{|x|}{\beta}}$, $x \in (-\infty, +\infty)$. Using the method of moments provide an estimator for β.

Answer to Problem 8.

So, in this case, we observe that we do not only take the first k moments when there are k parameters that we are estimating. Instead, we need to take moments that are a function of the parameters. In the above case, the first moment of X ended up being a constant (not a function of β) so we had to take the second moments and equate them.