Lecture 8 Worksheet

Chrysafis Vogiatzis

Every worksheet will work as follows.

1. You will be asked to form a group with other students in the class: you can make this as big or as small as you'd like, but groups of 4-5 work best.
2. Read through the worksheet, discussing any questions with the other members of your group.
   - You can call me at any time for help!
   - I will also be interrupting you for general guidance and announcements at random points during the class time.
3. Answer each question (preferably in the order provided) to the best of your knowledge.
4. While collaboration between students is highly encouraged and expected, each student has to submit their own version.
5. You will have 24 hours (see gradescope) to submit your work.

Activity 1: Exponential, Poisson, and Erlang

A manufacturing process requires the completion of 4 small steps: pre-processing, processing, inspection, packaging. Each step of the process requires time that is that is exponentially distributed with a rate of 1 completion every 3 minutes. That is, all steps are exponentially distributed with $\lambda = 1/3$ minutes. The steps have to be performed sequentially. Answer the following questions.

Problem 1

What is the probability that the "pre-processing" step alone (that is, **the first step alone**, from its start to its end) is completed within 4 minutes? [1]

Answer to Problem 1.

[1] Is the time in which a step, any step, is completed Poisson, exponential, or Erlang? Based on your answer, does it matter if you were asked about any other step or would your answer stay the same?

Problem 2

An inspector shows up to watch the operations take place. They only have time to be there for 10 minutes. What is the probability that there are exactly 3 steps that are completed in the 10 minutes after the inspector is there? [2]

[2] First, calculate the rate of completed steps in 10 minutes; then decide if you are using Poisson, exponential, or Erlang..

Answer to Problem 2.

Problem 3

What is the probability that a manufacturing order is completed within 10 minutes (all 4 steps, one after the other)? [3]

[3] Feel free to use an online calculator for your integral if you need to.

Answer to Problem 3.

We can answer Problem 3 using random variable T for the time until 4 steps are completed: T is Erlang with $k = 4$ steps and $\lambda = 1/3$ minutes. We could also calculated this using random variable X for the number of steps that are completed in 10 minutes: X is a Poisson random variable with $\lambda = 10/3$. Then, we can show that:

$$P(T \leq 10 \text{ minutes}) = P(X \geq 4 \text{ steps}).$$

Activity 2: The normal distribution

In this part of the worksheet, we turn our focus to the normal distribution. For this next part, we assume that $\Phi(z)$ is the standard normal distribution cumulative function. We can use the z-table provided in the last page to find $\Phi(z)$!

Problem 4: Converting to z values

Let's practice with converting to the proper z values. Let X be a normally distributed random variable with $\mu = 10, \sigma^2 = 4$.

Answer to Problem 4.

- $X = 12 \implies z =$
- $X = 8 \implies z =$
- $X = 4 \implies z =$

Problem 5: Simple normal distribution probabilities

For the previous random variable $X \sim \mathcal{N}(10, 4)$, find the probabilities. Use the z values you calculated earlier. [4]

Answer to Problem 5.

- $P(X \leq 12) =$
- $P(X \geq 4) =$
- $P(4 \leq X \leq 12) =$

[4] A z-table as described in the lecture notes is provided in the last page of the worksheet. Also recall that $\Phi(-z) = 1 - \Phi(z)$ due to symmetry.

Problem 6: Interesting probabilities

As we saw in class, the normal distribution is centered at μ. [5] A follow-up question would be to find the range of values centered at μ that satisfy a certain probability. Let's see an example here: what is the probability that X is within 1 unit from its center (μ), that is what is the probability that X is between 9 and 11? How about 2 units from its mean?

[5] So, following the previous random variable X, it would be centered at 10.

Answer to Problem 6.

- $P(9 \leq X \leq 11) =$

- $P(8 \leq X \leq 12) =$

Problem 7

Let's now focus on the opposite problem. What should the range be (centered at μ) so that the probability of the range is 50%? In essence, what should a be in order for $P(\mu - a \leq X \leq \mu + a) = 0.5$? Remember that earlier we calculated two range probabilities:

1. $P(9 \leq X \leq 11) = 0.383$.
2. $P(8 \leq X \leq 12) = 0.6826$.

Based on that, we should anticipate a to fall somewhere above 1 unit but below 2 units. But how big should it be exactly? Recall that we assume that $X \sim \mathcal{N}(10, 4)$. [6]

[6] $\mu = 10, \sigma^2 = 4 \implies \sigma = 2$.

Answer to Problem 7.

Problem 8

For $X \sim \mathcal{N}(10, 4)$, we want to see how big the range should be in order to have a probability equal to 95% that X falls in that range. In essence, we are now interested in $P(\mu - a \leq X \leq \mu + a) = 0.95$. Graphically:

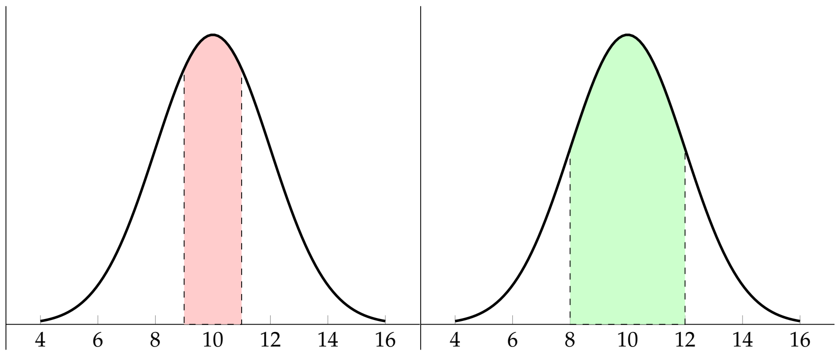
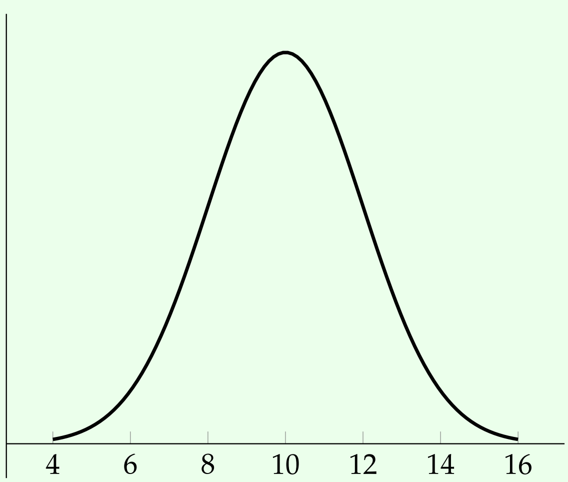


Figure 1: What should a be for $P(\mu - a \leq X \leq \mu + a) = 0.95$? Here we show in red the area for $P(\mu - 1 \leq X \leq \mu + 1) = 0.383$ and in green the area for $P(\mu - 2 \leq X \leq \mu + 2) = 0.6826$. It should make sense that $P(\mu - a \leq X \leq \mu + a) = 0.95$, $\mu - a$ and $\mu + a$ have to be points located even farther from the center μ.

Answer to Problem 8.

After you have found a, try to draw the resultant area!



Problem 9

We may have started observing that a does not depend on μ all that much. Instead it depends on σ. For example, seeing as we may write $P(\mu - a \leq X \leq \mu + a) = 0.5$ as a probability of z as follows:

1. $z_1 = \frac{\mu - \alpha - \mu}{\sigma} = -\frac{\alpha}{\sigma}$
2. $z_2 = \frac{\mu + \alpha - \mu}{\sigma} = \frac{\alpha}{\sigma}$.
3. Note that $z_1 = -z_2$.

Based on that: $P(\mu - a \leq X \leq \mu + a) = P(z_1 \leq Z \leq z_2)$. Now recall that $P(z_1 \leq Z \leq z_2) = \Phi(z_2) - \Phi(z_1) = \Phi(z_2) - \Phi(-z_2) = \Phi(z_2) - (1 - \Phi(z_2)) = 2\Phi(z_2) = 2\Phi(\frac{\alpha}{\sigma}) - 1$.

With that in mind, answer the following three questions. Try to answer them generally, not only for $X \sim \mathcal{N}(10, 4)$.

Answer to Problem 9.

- $P(\mu - \sigma \leq X \leq \mu + \sigma) =$
- $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) =$
- $P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) =$

Based on our answers, we have the following realization: σ is very important. Any normally distributed random variable probability can be expressed as a "distance" in terms of "σ". In essence:

$$\boxed{x = \mu + z\sigma \implies F(x) = \Phi(z).}$$

See this link for an interesting mnemonic, called the **68-95-99.7 rule**:

https://en.wikipedia.org/wiki/68–95–99.7_rule

Activity 3: Contrasting exponentials

Consider two exponentially distributed random variables X_1, X_2 with rates λ_1, λ_2.

Problem 10

What is the probability of $X_1 > X_2$, given that $X_2 = x$? [7]

[7] Can't we say that $P(X_1 > X_2|X_2 = x)$ is simply $P(X_1 > x)$?

Answer to Problem 10.

$P(X_1 > X_2|X_2 = x) = P(X_1 > x) =$

Problem 11

What is the probability of $X_1 > X_2$, in general? Recall the total probability law? [8] We can apply this to continuous distributions, too! We cannot sum here, but we may integrate. Let X_1 be random variable distributed with pdf $f(\cdot)$ and X_2 be a random variable distributed with pdf $g(\cdot)$, then:

[8] For an event B, and m mutually exclusive and collectively exhaustive events A_i, $i = 1, \ldots, m$, then we have $P(B) = \sum_{i=1}^{m} P(B|A_i) \cdot P(A_i)$.

$$\boxed{P(X_1 > X_2) = \int_{-\infty}^{+\infty} P(X_1 > X_2|X_2 = x)g(x)dx.}$$

Use this to answer the following question:

Answer to Problem 11.

$P(X_1 > X_2) =$

From this last part, we see that for two exponentially distributed random variables X_1, X_2 with rates λ_1, λ_2, respectively, we have:

$$\boxed{P(X_1 \leq X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}}$$

Problem 12

Two employees are using a website to place an order with a supplier at exactly the same time. The first person is more tech savvy and completes an order with rate 1 order every 3 minutes. The second person is just starting the job and learning, so they are a little slower and complete an order with rate 1 order every 5 minutes. Both times are exponentially distributed. What is the probability that the second person completes the order faster than or equal to the time the first person takes to complete an order?

Answer to Problem 12.

As a summary of the exponential distribution:

1. If the time between events is exponentially distributed, then the time until the k-th event ($k > 1$) is Erlang distributed, and the number of events within some time is Poisson distributed.

   Customers arrive with time that is exponentially distributed with rate $\lambda = 3$ customers every hour. Then:

   - "What is the probability the next customer shows up in 10 minutes?" is an exponential distribution question.
   - "What is the probability there are more than one customer in the next 20 minutes" is a Poisson distribution type of question. Recall that we will update λ to be in 20 minute intervals $\implies \lambda = 1$ customer every 20 minutes.
   - "What is the probability the second customer shows up within 20 minutes" is an Erlang distribution type of question.

2. The exponential distribution is **memoryless**! Hence:

$$P(T > s + t | T > s) = P(T > t).$$

3. If $X_1, X_2, \ldots, X_k$ are independent exponentially distributed random variables with rates $\lambda_1, \lambda_2, \ldots, \lambda_k$ respectively, then the probability X_i is the smallest one (i.e., $P(X_i < X_1, X_i < X_2, \ldots)$) is:

$$\frac{\lambda_i}{\lambda_1 + \lambda_2 + \ldots + \lambda_k}.$$

NORMAL CUMULATIVE DISTRIBUTION FUNCTION ($\Phi(z)$)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000