Lecture 9 Worksheet

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Every worksheet will work as follows.

1. You will be asked to form a group with other students in the class: you can make this as big or as small as you'd like, but groups of 4-5 work best.
2. Read through the worksheet, discussing any questions with the other members of your group.
   - You can call me at any time for help!
   - I will also be interrupting you for general guidance and announcements at random points during the class time.
3. Answer each question (preferably in the order provided) to the best of your knowledge.
4. While collaboration between students is highly encouraged and expected, each student has to submit their own version.
5. You will have 24 hours (see gradescope) to submit your work.

Activity 1: Basic expectation and variance properties

First, let's practice some basic expectation and variance properties. Answer the following questions. If False, correct the statement. [1]

[1] For these statements, consult with pages 3-4 and 8-9 of the notes.

Problem 1: True or False

Answer to Problem 1.

$$E\left[3X+2\right] = 3E\left[X\right]$$

a) True b) False

$$E\left[3X^2\right] = 3E\left[X^2\right]$$

a) True b) False

$$Var\left[3X+2\right] = 9Var\left[X\right] + 4$$

a) True b) False

For the next one, you may assume that X and Y are independent random variables.

$$Var\left[3X+Y\right] = 9Var\left[X\right] + Var\left[Y\right]$$

a) True b) False

Activity 2: Rating the latest Thor movie

A movie magazine decides to allow its reviewers to provide a rating ranging from 1 to 4 stars. Here, we are specifically focusing on Thor: Love and Thunder, the latest blockbuster in the Marvel Cinematic Universe. In the next two problems, we will see what the expected stars the movie will get if reviewers are allowed to provide an integer number of stars, or when they are allowed to providen *any real number* of stars.

Problem 2: Integer stars

Assume that the number of stars for any movie is represented as a **discrete random variable** X with pmf equal to $p(x) = \frac{x^2}{30}$ for $x = 1, 2, 3, 4$. What is the expected number of stars for any movie (that is, what is the expectation of X)? What is the variance of X? [2]

[2] For the variance, recall that you may use the formula

$$Var[X] = E\left[X^2\right] - (E[X])^2.$$

Answer to Problem 2.

$E[X] =$

$Var[X] =$

Problem 3: Real stars

Now, assume that the number of stars (again, from 1 to 4) a reviewer can give to a movie can be represented as a **continuous random variable** X with pdf equal to $f(x) = \frac{x^2}{21}$ for $1 \leq x \leq 4$. [3] What is the expected value of X in this case? How about the variance of X? [4]

[3] This means that a reviewer may opt to give a movie 3.5 stars, while another may decide to give out 2.78554 stars.

[4] The same variance formula $Var[X] = E[X^2] - (E[X])^2$ holds for both continuous and discrete random variables.

Answer to Problem 3.

$E[X] =$

$Var[X] =$

Activity 3: The law of ***total expectation***

In this activity, we derive the law of total expectation for both discrete random variables (in the form of a summation) and continuous random variables (in the form of an integral).

Problem 4: A simple case

Let us begin with something simple. An experiment is successful 90% of the time (and failed the remaining time). We perform 10 experiments. How many should we expect to be successful? [5]

[5] Think about what distribution this could be modeled as. Then, you may use the expectation formula for that specific distribution!

Answer to Problem 4.

Problem 5: External conditions

Let us complicate this slightly. Once again we perform 10 experiments, where an experiment can be successful or failed. However, the success probability depends on some external conditions. If the conditions are good, the probability of success is 95%; in average conditions, the probability becomes 90%; in bad conditions, the probability is lower at 75%. [6] All experiments will take place at the same conditions; so either all experiments will have good conditions, or all experiments will have average conditions, or all experiments will have bad conditions.

[6] Think of it like that: if the conditions are good, then the expected number of successes would be $0.95 \cdot 10 = 9.5$; if the conditions are average, then the expected number of successes would be $0.90 \cdot 10 = 9$; finally, if the conditions are bad, then the expectation becomes $0.75 \cdot 10 = 7.5$. Could we multiply each expectation with its respective probability? Are we allowed to do that?

Assuming conditions are equally probable (that is, good/average/bad conditions appear $\frac{1}{3}$ of the time), what is the expected number of successful experiments now?

Answer to Problem 5.

Problem 6: Generalizing the result

Can we generalize the previous result? What if we had m different possible conditions, each appearing with probability $\pi_i, i = 1, \ldots, m$ and each leading to probability of success p_i? How many experiments should we expect to be successful if we perform $n = 10$ experiments?

Answer to Problem 6.

Based on your answers so far, we observe that if we can partition the space in m mutually exclusive and collectively exhaustive events A_i each with probability of appearing equal to $P(A_i)$ [7], then the expected value of random variable X can be found by:

$$E[X] = \sum_{i=1}^{m} E[X|A_i] \cdot P(A_i)$$

How do you think this should look like for continuous random variables? [8]

[7] Look at this! This is also the setup for the law of total probability (see Lecture 4).

[8] Recall during the previous lecture we saw that summations become integrations, and probabilities become probability distribution functions..

Problem 7

Consider a continuous random variable with pdf $f(x) = \frac{1}{2}(1 + \theta \cdot x)$ for $-1 \leq x \leq 1$, where θ is uniformly distributed between 0 and 1. What is the expected value of X?

Answer to Problem 7.

The law of total expectation applies to continuous random variables, too. Consider X, Y as continuous random variables, such that we know $E[X|Y = y]$. Also assume that Y has pdf $g(y)$. Then, we have:

$$E[X] = \int_{-\infty}^{+\infty} E[X|Y = y] \cdot g(y)dy$$

Activity 4: A printer replacement policy

An office has bought a new printer which is supposed to have lifetime that is **exponentially distributed** with an expected lifetime at 2 years. [9] Answer the following questions.

[9] Usually you were provided rates for exponential distributions: however, we may now equivalently provide the expectation, since we know that for an exponentially distributed random variable X, its expectation $E[X]$ is $\frac{1}{\lambda}$, where λ is the rate.

Problem 8

What is the probability that the printer still works after 2 years?

Answer to Problem 8.

Problem 9

Based on your answer in Problem 8, if the company buys $n = 30$ printers for their 30 offices, how many of them should we expect to still work after 2 years? [10]

[10] To answer this question, consider what type of distribution fits the question.. It is as if we have $n = 30$ "tries" for printers to "survive for two years"..

Answer to Problem 9.

Problem 10

The company is starting a new policy. They will replace the printer either when it breaks down (recall that it breaks down in time that is exponentially distributed with like in Problem 8), or when it becomes 2 years old, whichever comes first. What is the expected lifetime of every printer the company buys? [11]

Note that if we allowed a printer to work until it breaks down, then we would (on average) expect to replace a printer every two years – as this is the given expectation of the exponential distribution. But if we replace any printer the moment they are 2 years old, then the expectation should go down, shouldn't it?

[11] An equivalent question: every how often does the company buy a new printer?

Answer to Problem 10.

Activity 5: Expectations of functions

This is optional. I will provide answers to this question before the exam. That said, if you do attempt it successfully, I will consider it for extra credit in the first exam.

Problem 11

Consider again the printer from Activity 4. The speed with which the printer works (and prints documents) is a function of its age. When the printer is x years old, its speed is given by $g(x) = \frac{\sqrt{x+1}}{x+1}$ velocity units. For example, when it is just bought, and the printer is 0 years old, its speed is equal to 1 velocity unit; on the other hand, after 2 years its speed drops to $\sqrt{3}/3$ velocity units. [12]

[12] If it helps ground the question better, you may think of 1 velocity unit as the biggest speed there is, and 0 velocity units as the lowest speed there is.

What is the expected speed of a printer if the company implements the policy of replacing the printer when it breaks down or when it becomes 2 years old, whichever comes first?

Answer to Problem 11.