Hypothesis testing for means and variances

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Lecture 26-27

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Overview

Hypothesis tests

On the mean
- Normal, known $\sigma$
  - Z statistic
- Normal, unknown $\sigma$
  - t statistic
- Non-normal, large sample
  - Z statistic

On the variance
- Normal distribution
  - $\chi^2$ statistic

On the proportion
- Non-normal, large sample
  - Z statistic
Null hypothesis: $H_0 : p = p_0$.

Test statistic: $Z_0 = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$.

Distribution: $Z_0 \sim \mathcal{N}(0, 1)$.

<table>
<thead>
<tr>
<th>$H_1$</th>
<th>Rejection region</th>
<th>$P$-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \neq p_0$</td>
<td>$</td>
<td>Z_0</td>
</tr>
<tr>
<td>$p &gt; p_0$</td>
<td>$Z_0 &gt; z_\alpha$</td>
<td>$1 - \Phi(Z_0)$</td>
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<td>$p &lt; p_0$</td>
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Reject if $Z_0$ or $\hat{p}$ falls in the rejection region or if $P$-value $< \alpha$. 
Hypothesis testing for means

Hypothesis tests

On the mean
- Normal, known $\sigma$
  - $Z$ statistic
- Normal, unknown $\sigma$
  - $t$ statistic
- Non-normal, large sample
  - $Z$ statistic

On the variance
- Normal distribution
  - $\chi^2$ statistic

On the proportion
- Non-normal, large sample
  - $Z$ statistic
Null hypothesis: \( H_0 : \mu = \mu_0 \).

Test statistic: \( Z_0 = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \).

Distribution: \( Z_0 \sim N(0,1) \).

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Reject if \( Z_0 \) or \( \bar{X} \) falls in the rejection region or if \( P \)-value < \( \alpha \).
Hypothesis testing for means of normally distributed populations with unknown variance

Null hypothesis: \( H_0 : \mu = \mu_0 \).

Test statistic: \( T_0 = \frac{\bar{X} - \mu_0}{s / \sqrt{n}} \).

Distribution: \( T_0 \sim T_{n-1} \).

Rejection region:

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Reject if \( T_0 \) or \( \bar{X} \) falls in the rejection region or if \( P \)-value < \( \alpha \).
Hypothesis testing for means of not normally distributed populations

Null hypothesis: \( H_0 : \mu = \mu_0 \).

Test statistic: \( Z_0 = \frac{\bar{X} - \mu_0}{s / \sqrt{n}} \).

Distribution: \( Z_0 \sim \mathcal{N}(0, 1) \).

Rejection region

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Reject if \( Z_0 \) or \( \bar{X} \) falls in the rejection region or if \( P \)-value < \( \alpha \).
Hypothesis testing for variances

- On the mean
  - Normal, known $\sigma$
    - $Z$ statistic
  - Normal, unknown $\sigma$
    - $t$ statistic
  - Non-normal, large sample
    - $Z$ statistic

- On the variance
  - Normal distribution
    - $\chi^2$ statistic

- On the proportion
  - Non-normal, large sample
    - $Z$ statistic
Hypothesis testing for variances of normally distributed populations

Null hypothesis: \( H_0 : \sigma^2 = \sigma_0^2. \)

Test statistic: \( \chi_0^2 = \frac{(n-1) s^2}{\sigma_0^2}. \)

Distribution: \( \chi_0^2 \sim \chi^2_{n-1}. \)

Rejection region:
- \( \sigma^2 \neq \sigma_0 \):
  - \( \chi_0^2 > \chi^2_{\alpha/2,n-1} \)
  - \( \chi_0^2 < \chi^2_{1-\alpha/2,n-1} \)

CI region:
- \( \sigma^2 > \sigma_0 \):
  - \( \left[ \frac{(n-1)\sigma^2_0}{\chi^2_{\alpha/2,n-1}}, \frac{(n-1)\sigma^2_0}{\chi^2_{1-\alpha/2,n-1}} \right] \)
- \( \sigma^2 < \sigma_0 \):
  - \( \left( -\infty, \frac{(n-1)\sigma^2_0}{\chi^2_{1-\alpha,n-1}} \right] \)

Reject if \( \chi_0^2 \) or \( \sigma^2_0 \) falls in the rejection region.
Example

A call center is concerned that call durations for a customer service representative are too erratic: high variations in call durations can lead to customer dissatisfaction who have to wait longer for a resolution. The company has collected data from \( n = 24 \) randomly selected phone calls from that specific customer representative and calculated that \( s = 5 \) minutes.

1. **Is there enough evidence to suggest that \( \sigma = 4 \) minutes? Use \( \alpha = 0.05 \).**

2. **Assume that we do not care about the standard deviation being lower than 4 minutes; instead, we are only interested if the standard deviation is higher than that. Is there enough evidence to suggest that \( \sigma = 4 \) minutes or is it higher than that? Again, you may use that \( \alpha = 0.05 \).**
Solution to the example

First, set up your hypothesis:

\[ H_0 : \sigma^2 = 16 \]
\[ H_1 : \sigma^2 \neq 16. \]

- Calculate \( \chi^2_0 = \frac{(n-1)s^2}{\sigma^0} = \frac{23 \cdot 5^2}{16} = 35.94. \)
- Find the critical values for \( \chi^2_{0.025,23} \) and \( \chi^2_{0.975,23} \) as 38.076 and 11.689, respectively.
- **Fail to reject** as \( \chi^2_{0.975,23} \leq \chi^2_0 \leq \chi^2_{0.025,23}. \)

For the second part, set up the hypothesis as:

\[ H_0 : \sigma^2 = 16 \]
\[ H_1 : \sigma^2 > 16. \]

- The test statistic is still \( \chi^2_0 = 35.94. \)
- However now we are only looking for \( \chi^2_{\alpha,n-1} = \chi^2_{0.05,23} = 35.172. \)
- **Reject** then as \( \chi^2_0 > \chi^2_{0.05,23}. \)
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